impact math
focus on the future

Measurement

Ontario Ministry of Education and Training
Queen's Park, Mowat Block,
Toronto, Ontario
M7A 1L2

Impact Math is a professional development program to help teachers of Grades 7/8 implement the new Mathematics curriculum. The program was developed by the Impact Math team at the Ontario Institute for Studies in Education of the University of Toronto (OISE/UT). The development of this resource document was funded by the Ontario Ministry of Education and Training. This document reflects the views of the developers and not necessarily those of the Ministry.

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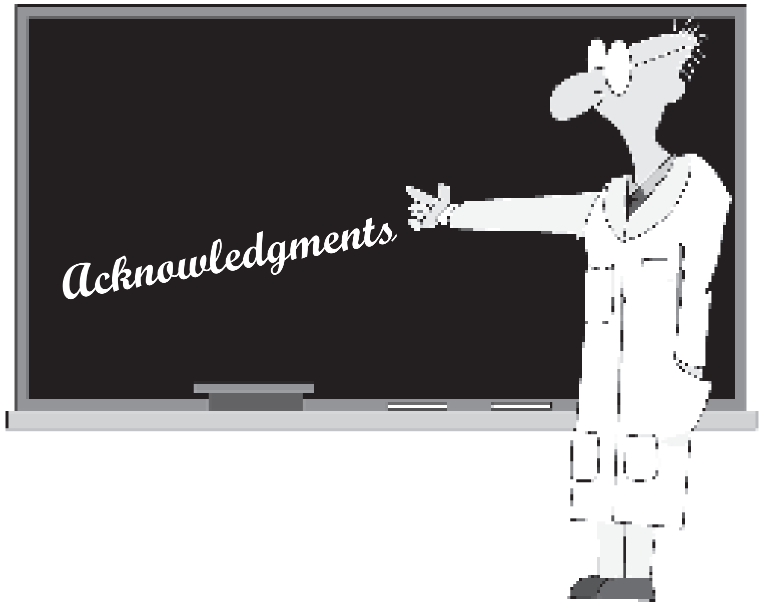

This module is the final product in a series of drafts, revisions and field tests conducted during the 1998–99 school year. Enhancing the usefulness of this booklet is the plethora of wonderful samples of student work that appear under the heading “WHAT YOU MIGHT SEE.” For these samples we are deeply indebted to the Grade 7 and 8 students of the following teachers:

Parklawn Junior Middle School
Toronto District School Board
Derek McLean

Johnsview Village Public School
York Region District School Board
Henry Lee

Bayview Hill Elementary School
York Region District School Board
David Valentim

Richmond Hill High School
York Region District School Board
Elizabeth Lehman

John G. Althouse Junior Middle School
Toronto District School Board
Adele Gibbins
Tricia Boyce
Bev Wright

Glen Shields Public School
York Region District School Board
Denise Hendrick
Frank Bellissmo

Islington Junior Middle School
Toronto District School Board
Steve Feldman

Don Mills Middle School
Toronto District School Board
Graham Beer
Marcia Schantz

We wish to thank these excellent teachers and their students for their invaluable contribution to this resource. Mike Wierzba and Barry Scully served as the liaisons who conducted the field tests, gathering information from students and teachers and providing insightful suggestions regarding the scoring guides, the activities, and the content of this module.

We acknowledge a significant debt to the following members of the *Impact Math* cadre of trainers who made important suggestions for improvements to the activities and the scoring guides to facilitate their use by teachers and strengthen their use as aids to report card preparation (see pp. 94–95).

Katie Branovacki	Trevor Brown	Jeri Lunney	André Mailloux	Pat Margerm
Lee McMenemy	Laura Romano	Caroline Rosenbloom	Jan Scully	Emilia Veltri

A special debt of gratitude is owed to Maria Bodianne, Sandra Dilena, Lorraine Schroetter, and Barry Scully who played a key role in the development of the scoring guides and the selection of student work.

The creation of such a document as this involves many stages of revision, rewriting, and reorganization, carrying with it a multitude of opportunities for errors. We are eternally grateful to Rosemary Tanner’s sharp eye and expert editorial skills for the many errors and omissions that she purged from the manuscript at each stage.

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Introduction to the Modules

The Ontario Curriculum, Grades 1– 8: Mathematics, issued in 1997, has redefined the elementary school mathematics curriculum for Ontario. New expectations for student learning require the teaching of new mathematical topics as well as a shift in emphasis of content previously taught. In particular, the new document reflects the growing need for students to expand their skills in processing information, managing data, problem solving, and using technology to achieve these ends. While there is a reduced attention to rehearsing rote skills, such as long division with large divisors or extraction of roots by the formal method, there is a reaffirmation of the need for students to master the multiplication tables and fundamental pencil-and-paper skills that underpin arithmetic facility. Such skills are intended to support the intelligent use of technology in performing complex computations of the type that arise in so-called "real world" contexts.

Implicit in this document is the demand for new or revised methods of instruction and assessment. Educational research of the past twenty years has mounted a compelling argument for a knowledge-building approach to instruction (see page 9) that reduces the role of the teacher as purveyor of information and enhances the teacher's role as facilitator of learning. With this shift in instructional methodology comes a corresponding demand for change in methods of assessment (see pages 10 – 12).

The call for such changes in curriculum, instruction, and assessment has created a need for teachers of grades seven and eight to plan new programs in mathematics from the plethora of print and electronic resources currently available. Since most of these teachers are responsible for many subject areas in addition to mathematics, the consolidation of these materials into a set of coherent lessons is daunting. To support teachers in this quest, the Ministry of Education and Training has commissioned a set of five modules (of which this is one) that gather together many of the extant resources in a single reference package. Each module addresses one of the five strands in the new curriculum.

Though they address different content strands, all modules have the same format. Part I outlines the rationale underpinning the ideas and activities developed in the module. Part II provides a brief instruction for teachers on the new content or approaches in that strand. Part III provides a set of four sample activities for Grade 7. Together these constitute an authentic task designed to consolidate and extend earlier developmental activities. This unit is intended to model the instructional and assessment philosophies discussed in Part I. **It is not intended to cover the entire content of the strand, nor to replace any resources presently used, but rather to supplement the current program.** Included in Part III under the heading "What You Might See" are samples of student work, classified by achievement level, and presented opposite a rubric that will help you assess the work of your students. Part IV parallels Part III, except it is keyed to the Grade 8 unit. However, it is recommended that all teachers familiarize themselves with the contents of both Parts III and IV. Part IV concludes with a selected list of appropriate print and media resources at the Grade 7–8 levels and some useful Internet addresses to fulfill the intent that the module provide a single reference to help teachers implement the new curriculum.

The Rationale For Measurement

Measurement

The rationale for the measurement strand in the Ontario mathematics curriculum is given on page 52 of *The Ontario Curriculum, Grades 1–8: Mathematics:*

> *Measurement concepts and skills are directly applicable to the world in which students live.*
>
> *The topic of measurement lends itself naturally to the introduction of fractions and decimals. It also requires students to be actively involved in solving and discussing problems. Students should be encouraged to compare objects directly by covering them with various units and counting the units. Concrete experience in solving measurement problems is the best foundation for using instruments and formulas. As students develop increasing skills in numeration, they can be challenged to undertake increasingly complex measurement problems, thereby strengthening their facility in both areas of mathematics.*
>
> *Estimation activities are important to help students become familiar with different measures and the process of measuring, and to gain an awareness of the size of units. Often, only an estimate is required in order to make a decision about the solution to a problem. Students should develop a fund of informal measurement guidelines, so that they know, for example, that their fingers are about 1 cm wide, that a can of pop is about 350 mL, and so on. These guidelines will help them estimate sizes in the world around them.*

The National Council of Teachers of Mathematics (NCTM) adds support to this vision by its inclusion of Measurement as Standard 13 in *Curriculum and Evaluation Standards for School Mathematics* (p. 118):

> *As students progress through grades 5–8, they should develop more efficient procedures and, ultimately, formulas for finding measures. Length, area, and volume of one-, two-, and three-dimensional figures are especially important over these grade levels. For example, once students have discovered that it is possible to find the area of a rectangle by covering a figure with squares and then counting, they are ready to explore the relationship between areas of rectangles and areas of other geometric figures. This exploration gives students an opportunity to reason deductively and see how mathematical ideas relate to one another. The following suggest some possibilities.*
>
> - *The area of a parallelogram can be rearranged into a rectangle.*
> - *The area of a triangle is one-half the area of a parallelogram.*
>
> *All these connections require students to understand that the area of a figure does not change if it is partitioned and rearranged. It is also important that students understand the association between multiplication and determining the area of a rectangle. The formula is not a "magic box." It is a summary of a process that tells how many units it takes to cover the rectangle. It is also a summary of the relationship among area, height, and length.*

In the sample unit presented in Part III of this document, students join Gulliver on his travels to Laputa, the land of the mathematicians, where they explore the relationships among the areas of triangles, rectangles, parallelograms, and trapezoids. In Part IV, they investigate the ingenious beverage can and calculate the area of its template as it is transformed into the surface area of a cylinder.

The Role of Technology in the Measurement Strand

The policy on the use of technology, as embodied in *The Ontario Curriculum, Grades 1–8: Mathematics,* is stated on page 7 of that document:

> *Students are expected to use calculators or computers to perform operations that are lengthier or more complex than those covered by the pencil-and-paper expectations. When students use calculators and computers to perform operations, they are expected to apply their mental computation and estimation skills in predicting and checking answers. Students will also use calculators and computers in various experimental ways to explore number patterns and to extend problem solving.*

The rationale for this policy is expressed clearly and strongly on page 17 of the *National Council of Teachers of Mathematics 1998–99 Handbook:*

> *Technology has changed the ways in which mathematics is used and has led to the creation of both new and expanded fields of mathematical study. Thus, the technology is driving change in the content of mathematics programs, in methods for mathematics instruction, and in the ways that mathematics is learned and assessed. A vital aspect of such change is a teacher's ability to select and use appropriate instructional technology to develop, enhance, and extend students' understanding and application of mathematics. It is essential that teachers continue to explore the impact of instructional technology and the perspectives it provides on an expanding array of mathematics concepts, skills, and applications.*

The position statement of the NCTM follows this rationale with six recommendations including the following two:

- *Every student should have access to an appropriate calculator.*
- *Every school mathematics program should provide students and teachers access to computers and other appropriate technology for individual, small-group, and whole-class use, as needed, on a daily basis.*

For the complete position statement including all six recommendations, access the NCTM's Web site at www.nctm.org/about or e-mail infocentral@nctm.org.

The *Measure* menu in dynamical geometric software such as *Geometer's Sketchpad* and *Cabri Geometry* provides students with a rich resource for exploring measurement concepts in a geometric context. The Grade 8 unit in Part IV of this module develops the relationship $C = \pi D$ between the circumference of a circle and its diameter using beverage cans, string, and measuring tape. An impressive way to reinforce and extend this concept is to have students draw a circle and its diameter in *Sketchpad* and watch the values of C, D, and C/D as they drag the circle and expand its diameter. The value of C/D remains constant while the values of C and D spin larger like the gauges on a gas pump. A similar activity can be used to show the invariance of the A/D^2 where A denotes the area of a circle of diameter D.

The unit in Part IV also employs the spreadsheet as a convenient tool for graphing C vs. D and A vs. D^2 to discover that the quotients C/D and A/D^2 have the same value for all circles. Activity 4 in that unit requires that students use the Internet to explore the conservation and recycling of aluminum beverage cans. Although calculators, computers, spreadsheets, and the Internet are invaluable in the measurement strand of the mathematics curriculum, they do not replace the need for student exploration with fundamental measuring devices like measuring tape, protractors, and grid paper.

Understanding the Learning Process & Its Impact on Instruction

In this and the other four modules, we present activities that attempt to incorporate a range of instructional approaches. The students are sometimes given information and required to read, interpret, and apply it in an exercise. In other cases, the students must investigate, explore, and discover concepts that lurk beneath the surface of an activity. In some cases, the students will work individually, while in others they will work collaboratively or cooperatively. The sample unit in Part III of this module confronts students with a classical area paradox that students, grouped in pairs, must explore, analyze, and resolve. Then the pairs of students must present and defend their conclusions in an open forum with their classmates. The sample unit in Part IV involves students in investigating ways to redesign, recycle, and reuse aluminum beverage cans to ensure that we do not exhaust the world supply of aluminum. Students are encouraged to work individually or in groups using scissors, measuring tape, centimetre paper, spreadsheets and (where available) dynamical geometry software as tools for exploration.

In view of these multiple perspectives on how children learn, one might assume that all traditional approaches to teaching will disappear as these philosophies are incorporated. However a response to the question “What should I see in a [NCTM] Standards-based mathematics classroom?” the *NCTM 1997–98 Handbook* presents a balanced and accessible image of effective instruction:

> *First and foremost, you'll see students doing mathematics. But you'll see more than just students completing worksheets. You'll see students interact with one another, use other resources along with textbooks, apply mathematics to real-world problems, and develop strategies to solve complex problems.*
>
> *Teachers still teach. The teacher will pose problems, ask questions that build on students' thinking, and encourage students to explore different solutions. The classroom will have various mathematical and technological tools (such as calculators, computers, and math manipulatives) available for students to use when appropriate. The teacher may move among the students to understand their thinking and how it is reflected in their work, often challenging them to engage in deeper mathematical thinking.*

Assessment: Rubrics & Achievement Levels

The changes in curriculum and instruction described on the preceeding pages have significant implications for assessment and evaluation. Among these implications is the shift from norm-referenced to criterion-referenced assessment, as described on page 1 of *The Assessment Standards for School Mathematics* published by the NCTM in 1995:

> *At present, a new approach to assessment is evolving in many schools and classrooms. Instead of assuming that the purpose of assessment is to rank students on a particular trait, the new approach assumes that high public expectations can be set that every student can strive for and achieve, that different performances can and will meet agreed-on expectations, and that teachers can be fair and consistent judges of diverse student performances.*

The Ontario Curriculum, Grades 1–8: Mathematics (see pp. 4–5) also embraces the move to criterion-referenced assessment and includes four levels of achievement for describing student performance:

> *High achievement is the goal for all students, and teachers, students, and parents need to work together to help students meet the expectations specified. The achievement levels are brief descriptions of four possible levels of student achievement. These descriptions, which are used along with more traditional indicators like letter grades and percentage marks, are among a number of tools that teachers will use to assess students' learning. The achievement levels for mathematics focus on four categories of skills: problem solving, understanding of concepts, application of mathematical procedures, and communication of required knowledge. When teachers use the achievement levels in reporting to parents and speaking with students, they can discuss with them what is required for students to achieve the expectations set for their grade.*

Descriptions of the four levels of achievement for problem solving, concepts, applications, and communication are shown on page 9 of that document. These are the levels for concept understanding:

knowledge/skills	Level 1	Level 2	Level 3	Level 4
Understanding of concepts	**The student shows understanding of concepts:**			
	– with assistance	– independently	– independently	– independently
	– by giving partially complete but inappropriate explanations	– by giving appropriate but incomplete explanations	– by giving both appropriate and complete explanations	– by giving both appropriate and complete explanations and by showing that he or she can apply the concepts in a variety of contexts
	– using only a few of the required concepts	– using more than half the required concepts	– using most of the required concepts	– using all of the required concepts

A table such as the one above that describes levels of achievement is called a *rubric*. Included with the student activities, in this and the other modules, are rubrics and samples of student work that exemplify the levels of student performance as defined in *The Ontario Curriculum, Grades 1–8: Mathematics.*

The release of the first module in this series, *Data Management & Probability*, was met with widespread enthusiasm. It confirmed our belief that teachers need and want materials to help them implement the new mathematics curriculum. Of particular interest to teachers are the issues associated with assessment and evaluation. The shift in emphasis from rote learning to higher-order processes, such as problem solving, drawing inferences, and communicating mathematical conclusions, requires that methods of performance assessment be added to the battery of devices that teachers use to assess mathematical learning. As observed in the NCTM publication *Curriculum and Evaluation Standards for School Mathematics: Addendum Series – A Core Curriculum* (1992):

> *Questions eliciting open-ended responses require more holistic approaches for scoring. Indirectly, they convey to students the need to communicate their ideas clearly and to construct their responses for a purpose. The impact on the curriculum of this type of assessment is to hold students accountable for demonstrating their understanding of connected ideas rather than displaying their proficiency with disconnected skills.* (p. 11)

One of the most important devices for the holistic scoring of higher-order tasks is the rubric. The rubric shown on page 10 is an example of what is called a "general rubric." In its publication *Assessment Standards for School Mathematics* (1995), the NCTM defines a general rubric as "an outline for creating task-specific rubrics" (p. 90). Furthermore it defines a "task-specific rubric" as a rubric that "describes levels of performance for a particular complex task and guides the scoring of that task consistent with relevant performance standards." In this module we present, under the heading WHAT YOU MIGHT SEE, samples of student responses to the activities. Large samples of student work collected during the field tests of these materials were used to create scoring guides. These guides are task-specific rubrics. You will notice however that they evaluate the "product", i.e., the student work, while the general rubric shown on page 10 includes an observational component of assessment (e.g., "with assistance," "independently"). Since there can be no observational component in the assessment of *completed* student work, the scoring guides in this book do not use phrases such as "independently" or "with assistance." **It is expected that teachers will use each scoring guide as a starting point in the development of a task-specific rubric that will evolve as it is used with students.**

On page 12, we offer some suggestions on how to develop task-specific scoring guides. However, it is important to recognize that the creation of rubrics is highly subjective and is more an art than a science. In the *TIMSS Monograph #1: Curriculum Frameworks for Mathematics and Science* (1993), Robitaille et al. issue this caveat:

> *Measuring educational achievement is difficult from both a conceptual and a practical perspective. What counts as "achievement" is not always easy to discern and even when a concept of achievement has been clearly explicated, ways and means for assessing it are not easily devised. The ongoing debate about educational measurement and the increasing number of alternative assessment approaches proposed in educational circles attest to this problem.* (p. 36)

Some Suggestions for Creating Your Own Scoring Guides

There are no set rules for constructing scoring guides. Each teacher will have personal preferences and individual conceptions that contribute a significant subjective component to this assessment instrument. Consequently there will be some variation among teachers in the levels of achievement assigned to a particular student response. However, the process described below presents the main elements in constructing scoring guides that many educators have found effective.

❶ If there are other teachers who are teaching mathematics at the same grade level, plan to set aside about 90 minutes to work together.

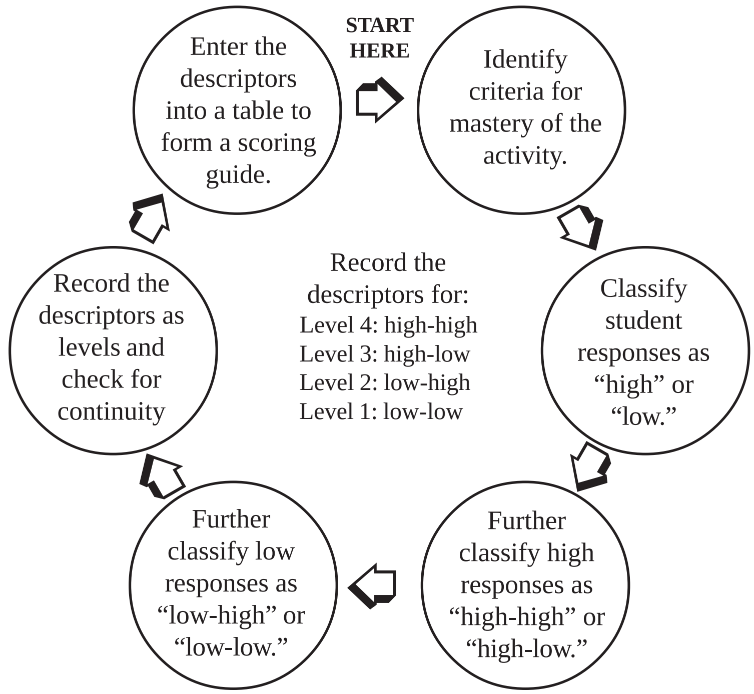
❷ Make a complete set of samples of student work on the activity for which you are developing the scoring guide. Distribute these to each colleague before the meeting

❸ Decide upon the kinds of responses that constitute mastery of the task. Identify responses that constitute various levels of partial mastery (about 20 minutes).

❹ Have each teacher individually assign the level "high" or "low" to each student response (about 20 minutes).

❺ When all student responses have been classified as "high" or "low," discuss and reach consensus on the rating of each response (about 20 minutes).

❻ Gather all the responses classified as "high" and repeat the process in ❹ and ❺, assigning the rating "high-high" or "high-low" to each of the "high" responses. Record descriptors used to determine each classification. Assign the "high-low" descriptors to "Level 3" and the "high-high" descriptors to "Level 4" (about 10 minutes).



Steps in Creating a Scoring Guide

❼ Repeat the procedure in ❻, classifying each of the "low" responses as "low-high" or "low-low." Reach consensus and record the descriptors. Assign the low-high descriptors to Level 2 and the low-low descriptors to Level 1 (about 10 minutes).

❽ Review the descriptors for the four levels to ensure that they form a continuum of increasing expectation and capture the criteria established in ❸. Record the descriptors in a scoring guide using a format such as in this book.

Helpful Resources for Creating Scoring Guides

Bryant, Deborah and Mark Driscoll. *Exploring Classroom Assessment in Mathematics.* Reston, VA: National Council of Teachers of Mathematics, 1998.

Danielson, Charlotte. *A Collection of Performance Tasks and Rubrics: Upper Elementary School Mathematics.* Larchmont, NY: Eye on Education, Inc., 1997.

Flewelling, Gary and Chuck Lemenchik. *Mathematics Assessment: Grades 7 & 8.* Toronto, Ontario: Gage Educational Publishing Company, 1997.

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PART II

What's New in Measurement?

Measurement: Perceptual Anchors & Estimation

New instructional approaches with an increased emphasis on estimation embody the major changes in the measurement strand. On page 7, we described briefly the rationale for the inclusion of measurement in the curriculum as stated in *The Ontario Curriculum, Grades 1– 8: Mathematics*. This rationale stressed the importance of estimation as a life skill.

An equally important impetus for the inclusion of a strong estimation component in the measurement strand derives from some insights in cognitive psychology. In the late 1960's, cognitive psychologist David Ausubel[1] observed,

> *If I had to reduce all educational psychology to just one principle, I would say this: The most important single factor influencing learning is what the learner already knows.*

Ausubel recognized that new knowledge must be built on the foundation of what the learner already understands. In the mid 1970's when Ontario teachers were involved in the conversion to metric units, Chester Carlow at OISE was applying Ausubel's insights to a program designed to foster the development of systematic estimation skills. These instructional materials pioneered the use of referents which he called *perceptual anchors*. Carlow[2] states,

> *In order to make direct estimates, children must internalize appropriate standard referents or perceptual anchors. A perceptual anchor is a quantity whose size is readily perceived in different settings; thus, a child can use internalized perceptual anchors to make comparative estimates of other quantities.* (p. 96)

This idea is further explicated in this paragraph (and particularly in the last sentence) from the NCTM publication, *Curriculum and Evaluation Standards for School Mathematics* (p. 117)

> *In everyday life, people need to make many kinds of measures to resolve common questions: About how long will it take? About how much do I need to buy? About how much will it hold? An estimate is often sufficient. Estimation requires a judgment about an entity's approximate relationship to a standard. Students' skills at estimating measurements will develop only through experience … The ability to hold one's hands about a meter apart, to know the length of a foot or stride, to know the width of a fingernail– all these are useful estimating tools.*

It is expected that students in Ontario will develop referents or perceptual anchors for standard measurement units as they progress through the elementary grades. However, by Grade 7, it is expected that they will use these perceptual anchors to approximate distances in metres, floor areas in square metres, volumes in millilitres, and so on. The activities in the Grade 7 sample unit in this module have students cutting out quadrilaterals from centimetre paper and calculating their areas in square centimetres. The sample unit in Grade 8 involves students in estimating in square centimetres the area of a circle by counting squares and parts of squares inside the circle. The formula for the area of a circle is developed in terms of its diameter rather than its radius i.e., $A = \pi D^2/4$ rather than πR^2, to help students establish the perceptual anchor of its area as approximately 3/4 the area of the square in which it is inscribed. As students formalize their understanding of such measurement concepts, they will develop increasingly more sophisticated ways of approximating and computing lengths, distances, areas, and volumes. Their facility with holistic estimation will enable them to assess whether their computations are yielding answers of the correct magnitude. This is the ultimate goal in developing number sense in measurement!

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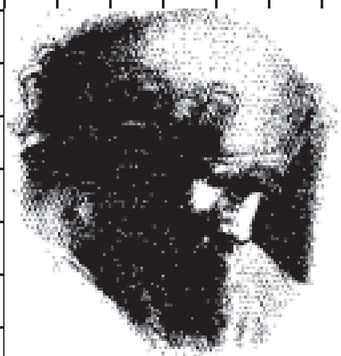
[1] Ausubel, David P. *Educational Psychology: A Cognitive View.* New York: Holt, Rinehart & Winston, 1968.

[2] Carlow, Chester D. "Critical Balances and Payoffs of an Estimation Program." In *Estimation and Mental Computation.* 1986 Yearbook of the National Council of Teachers of Mathematics, Reston, VA: 1986, pp. 93-102.

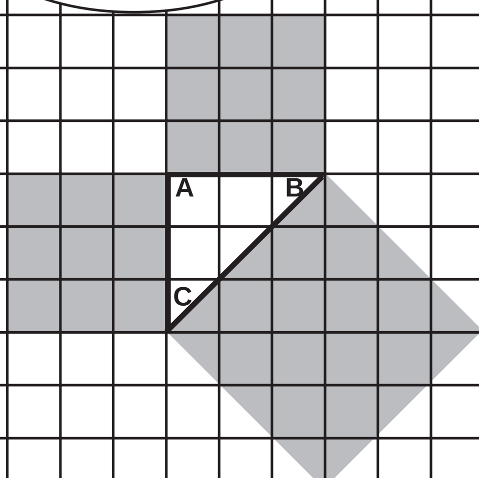
The Pythagorean Relationship



Pythagoras
582 B.C – 501 B.C.

The postage stamp shown here celebrates the Pythagorean relationship that was discovered over 2500 years ago. It shows a right triangle with a square drawn on each side of the triangle. Each small square has an area of one square unit. Count the small squares to record the area of the square on each side. Describe what you discover. This is true for all right triangles and is called the *Pythagorean relationship* or *Pythagorean theorem*. Named in honor of the mathematician, Pythagoras, of ancient Greece, it remains the most famous theorem in mathematics.

Count squares and half-squares to show that the Pythagorean relationship is true for ΔABC below.



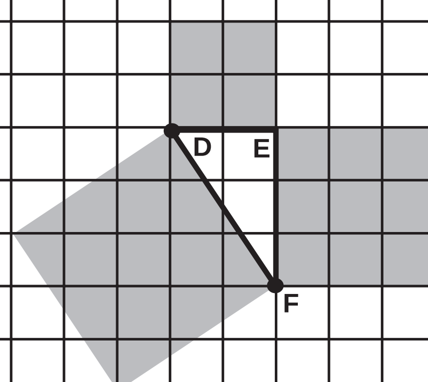
Use the area of the square on side BC of ΔABC to calculate the length of side BC.

Explain how you can use the lengths of the two shorter sides of a right triangle to calculate the length of the longest side (called the *hypotenuse)*.

Use the Pythagorean relationship to determine:

- the length of side DF.
- the distance between points D and F.

Explain how you can calculate the distance between two points on a grid by counting squares.

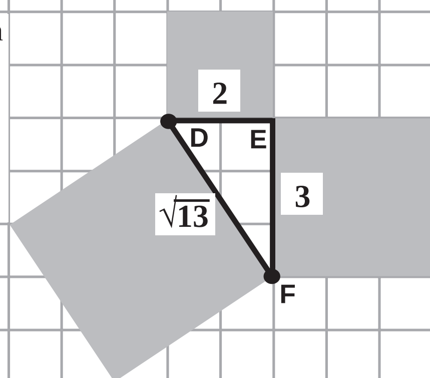


TEMPLATE

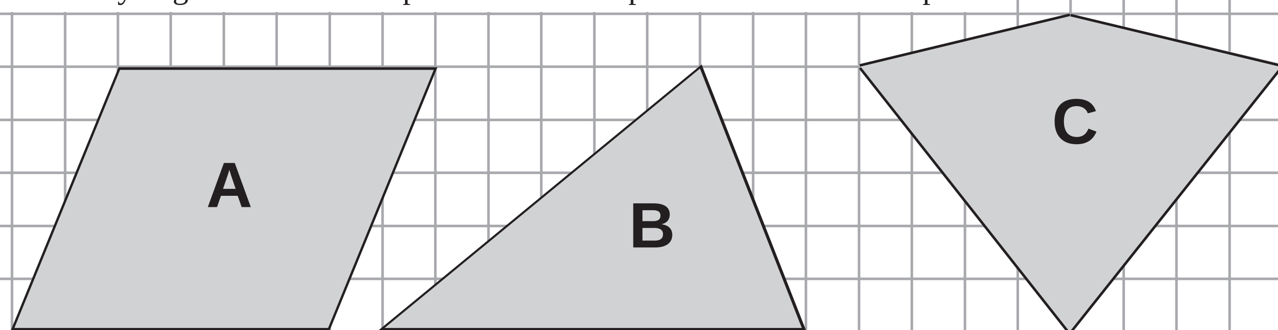
USING THE PYTHAGOREAN RELATIONSHIP TO CALCULATE PERIMETER

Finding the length of a line segment on a grid is easy when the segment is horizontal or vertical–you just count along the grid. When the segment is neither vertical nor horizontal, you can find its length by applying the Pythagorean relationship as shown below. **Take the distance between adjacent parallel grid lines as one unit.**

To calculate the length |DF| we write:
|DE| = 2 and |EF| = 3,
so $|DF|^2 = 2^2 + 3^2$
$= 13.$
$|DF| = \sqrt{13} \approx 3.60.$



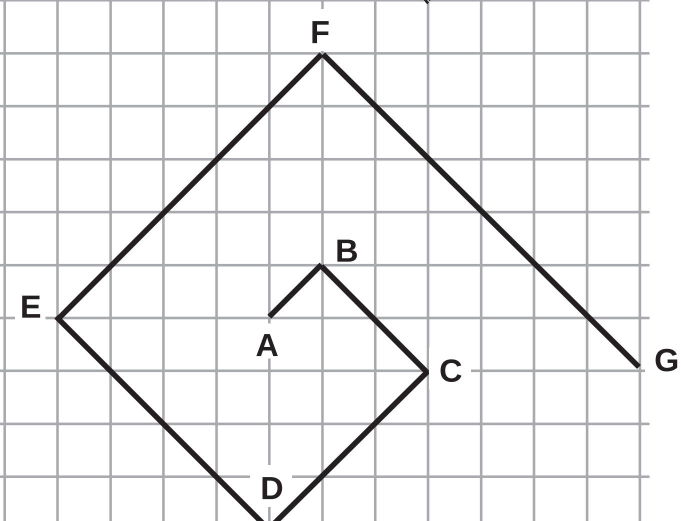
❶ Use the Pythagorean relationship to calculate the perimeters of these shapes.



❷ Estimate the length of the path
A → B → C → D → E → F → G.

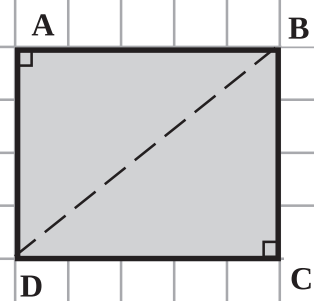
Then calculate the length of this path.

How close was your estimate? Explain how you could have calculated the length of the path using the Pythagorean relationship only once.

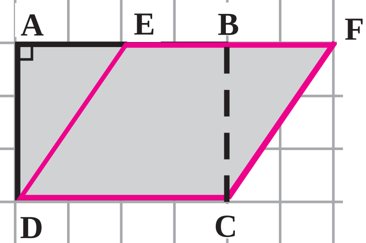


Relating the Area of a Parallelogram to the Area of a Rectangle

Take the distance between adjacent parallel grid lines as one unit. Therefore each small square has an area of one square unit.



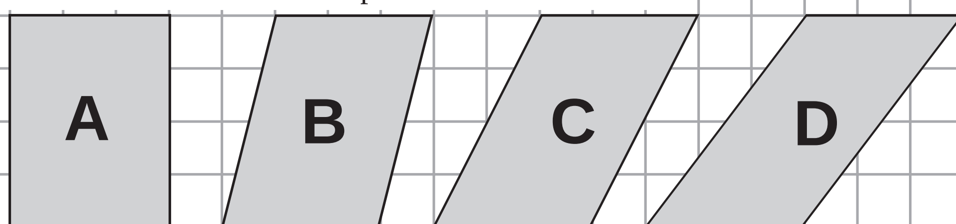
❶ Diagonal BD divides rectangle ABCD into two triangles, ΔABD and ΔCDB.
Are these two triangles congruent?
Do they have equal area? Explain how you know.
What are the areas of ΔABD and ΔCDB?
Explain how you can calculate the area of a right triangle on a grid.



❷ Rectangle ABCD and parallelogram EFCD share a common base DC.
What triangle is part of ABCD but is outside EFCD?
What triangle is part of EFCD but is outside ABCD?
What are the areas of those two triangles?
Compare the areas of ABCD and EFCD. Describe what you discover.

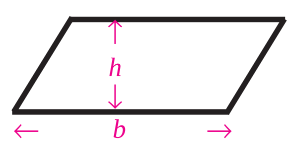
❸ Record the base, height, and area of each of these shapes in the table below.

Shape	Base	Height	Area
A			
B			
C			
D			



Look for a pattern in your table. Describe what you discover.

❹ a) Explain how to calculate the area of a parallelogram if you know the length *b* of its base and its height *h*.
b) Write a formula for the area A of a parallelogram in terms of *b* and *h*.



Challenge

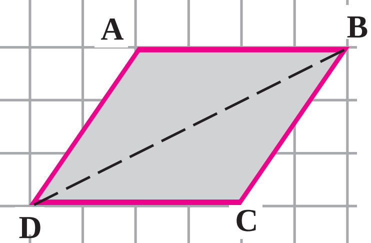
When can the area of a parallelogram be calculated by multiplying its length by its width?

TEMPLATE

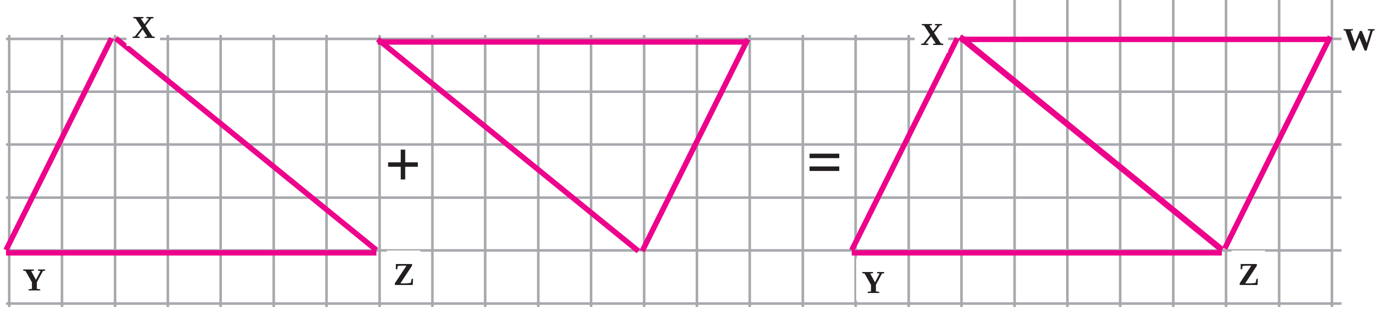
Relating the Area of a Triangle to the Area of a Parallelogram

Take the distance between adjacent parallel grid lines as one unit. Therefore each small square has an area of one square unit.

❶ Diagonal DB divides parallelogram ABCD into two triangles, ΔABD and ΔCDB.
Are these two triangles congruent?
Do they have equal area? Explain how you know.
What is the area of parallelogram ABCD?
What are the areas of ΔABD and ΔCDB?
Explain how you can calculate the area of a triangle formed by drawing the diagonal of a parallelogram.

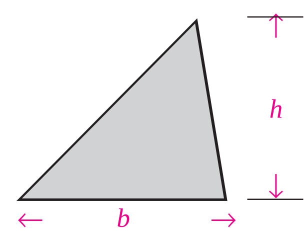


❷ One way to determine the area of ΔXYZ is to make a congruent copy and match them to form a parallelogram XYZW.



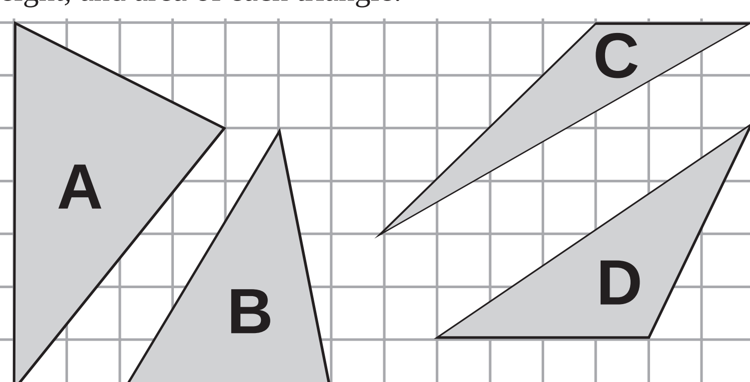
What is the area of parallelogram XYZW? What is the area of ΔXYZ? Explain how to calculate the area of a triangle on a grid *without* constructing a congruent copy.

❸ a) Explain how to calculate the area of a triangle if you know the length *b* of its base and its height *h*.
b) Write a formula for the area A of a triangle in terms of *b* and *h*.



❹ Record in the table below, the base, height, and area of each triangle.

Triangle	Base	Height	Area
A			
B			
C			
D			



PART III

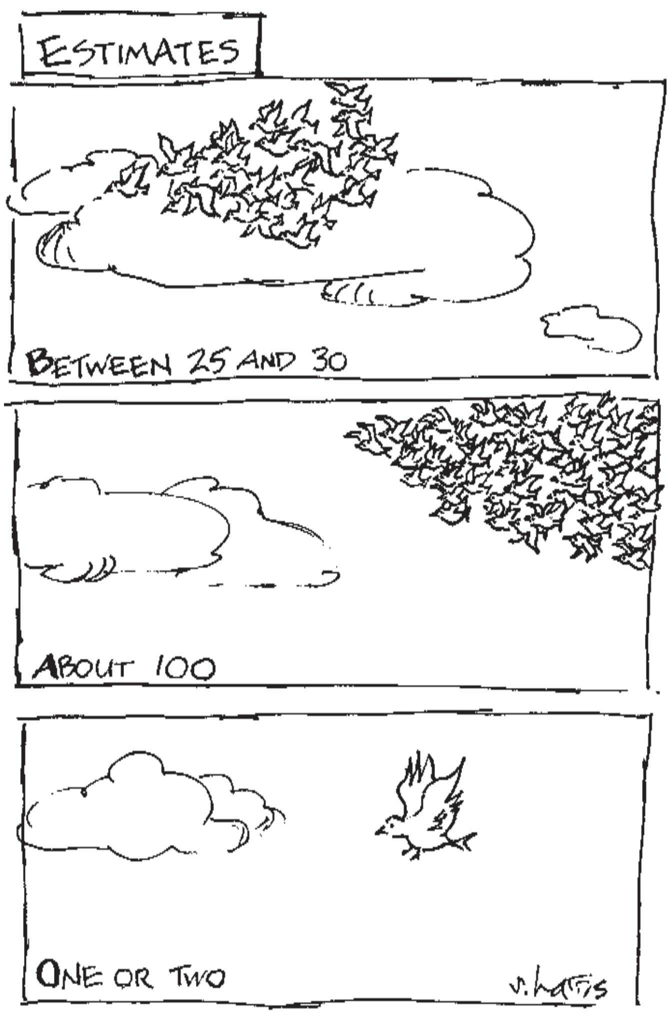
Measurement in Grade 7

Measurement: Grade 7

Overall Expectations

By the end of Grade 7, students will:

- demonstrate a verbal and written understanding of and ability to apply accurate measurement strategies that relate to their environment;
- identify relationships between and among measurement concepts (linear, square, cubic, temporal, monetary);
- solve problems related to the calculation and comparison of the perimeter and the area of irregular two-dimensional shapes;
- apply volume formulas to problem-solving situations involving rectangular prisms.



Specific Expectations

For convenient reference, the specific expectations are coded. M 7-1 refers to the first Measurement expectation in Grade 7.

Students will:

Units of Measure

M 7-1 - create definitions of measurement concepts;

M 7-2 - describe measurement concepts using appropriate measurement vocabulary;

M 7-3 - research and report on uses of measurement instruments in projects at home, in the workplace, and in the community;

M 7-4 - make increasingly more informed and accurate measurement estimations based on an understanding of formulas and the results of investigations;

Perimeter & Area

M 7-5 - understand that irregular two-dimensional shapes can be decomposed into simple two-dimensional shapes to find the area and perimeter;

M 7-6 - estimate and calculate the perimeter and area of an irregular two-dimensional shape (e.g., trapezoid, hexagon);

M 7-7 - develop the formula for finding the area of a trapezoid;

M 7-8 - estimate and calculate the area of a trapezoid, using a formula;

M 7-9 - draw a trapezoid given its area and/or perimeter;

M 7-10 - develop the formulas for finding the area of a parallelogram and the area of a triangle;

M 7-11 - develop the formula for finding the surface area of a rectangular prism using nets;

Capacity, Volume, and Mass

M 7-12 - develop the formula for finding the volume of a rectangular prism (area of base × height);

M 7-13 - understand the relationship between the dimensions and the volume of a rectangular prism;

M 7 -14 - calculate the surface area and the volume of a rectangular prism in a problem-solving context;

M 7-15 - sketch a rectangular prism given its volume.

Activity 1 – Teacher Edition

Gulliver Dines with the Mathematicians

Expectations Addressed

M 7-1 create definitions of measurement concepts.

M 7-2 describe measurement concepts using appropriate measurement vocabulary.

M 7-4 make increasingly more informed and accurate measurement estimations based on an understanding of formulas and the results of investigations.

M 7-5 understand that irregular two-dimensional shapes can be decomposed into simple two-dimensional shapes to find the area and perimeter.

M 7-6 estimate and calculate the perimeter and area of an irregular two-dimensional shape (e.g., trapezoid, hexagon).

M 7-8 estimate and calculate the area of a trapezoid, using a formula.

Context

The estimation and calculation of areas of triangles and quadrilaterals is embedded in the rich fantasy of *Gulliver's Travels*. This famous classic, written by Jonathon Swift around 1725, is a political satire in which Swift vents his somewhat cynical view of the human race. In Gulliver's visits to Lilliput and Brobdingnag, Swift combines his political commentary with an exploration of the relativity of size, and his satire is manifest mainly as humour. However, in Gulliver's visit to Laputa, Swift's satire transforms itself into an angry assault against mathematicians. (Apparently the triumph of science, evidenced by the widespread popularity of Isaac Newton's *Principia*, had challenged Swift's theological teachings, and theoretical mathematicians became the target of Swift's anger.)

In Activity 1, students are presented with a (slightly modernized) excerpt from *Gulliver's Travels* in which Gulliver describes his dinner in Laputa. Since Swift confuses 2-D shapes and 3-D figures, this activity invites students to rewrite one paragraph correcting the errors. Students also estimate and calculate perimeters and areas of triangles and quadrilaterals on a grid. This helps them review and consolidate the formulas for the areas of triangles and parallelograms called for at the Grade 6 level in *The Ontario Curriculum, Grades 1–8: Mathematics* (p. 38). In Activity 2, students help the mathematicians escape the King's new tax on right angles by transforming rectangles into trapezoids of equal area. This leads them to discover inductively the formula for the area of a right-angled trapezoid. Activity 3 builds on the previous explorations and formalizes the area formula for the trapezoid. Finally, in Activity 4, students are thrust into the role of tax appraiser to confront the classical paradox of the four shapes of total area 64 square units that seem to fit together to form a rectangle of area 65 square units.

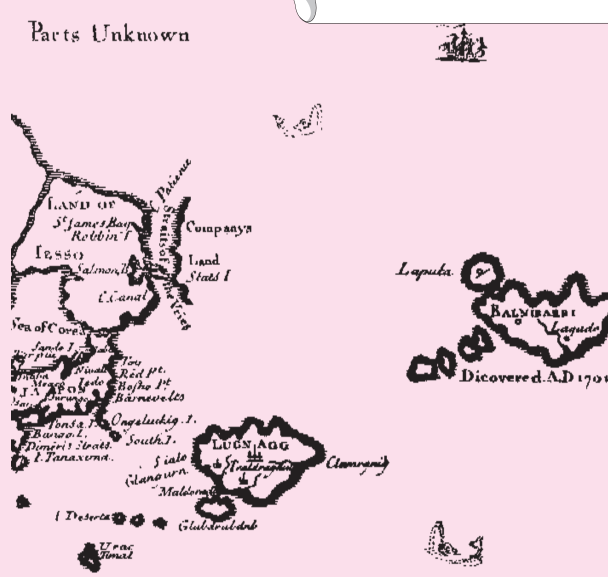
Activity 1 – Student Page

Gulliver Dines with the Mathematicians

Gulliver's Travels is a popular tale of a traveller named Gulliver who sailed the oceans to strange and distant lands. Most people know of his visit to Lilliput, the land of the little people. Some know of his visit to Brobdingnag, island of the giants. But few have read the chapter about Gulliver's visit to Laputa – the land of the mathematicians. Some small excerpts from that visit are presented here in a slightly modified form, to modernize the old English in which this manuscript was written almost three centuries ago!

We had two courses of three dishes each. In the first course, there was a shoulder of mutton, cut into an equilateral triangle; a piece of beef into a rhombus and a pudding into a cycloid. … The servants cut our bread into cones, cylinders, parallelograms and several other mathematical figures.

…Their ideas are perpetually expressed in lines and figures. To praise the beauty of an animal, they describe it in terms of rhombuses, circles, parallelograms, ellipses and other geometric terms.



The Lesson Launch
10 minutes

The intent of this lesson is to have students learn the distinctions between 2-D shapes and 3-D figures, to explore the concepts of perimeter and area, and to review the formulas for the areas of triangles and quadrilaterals.

To launch this lesson, ask students what they know (if anything) of *Gulliver's Travels*. Provide students with a brief overview of Gulliver's trips to Lilliput and Brobdingnag. Then describe the visit to Laputa. (You may enhance your presentation by making overhead transparencies of the illustrations included in this unit or from Asimov (see p. 96).

Since some students will have difficulty reading the excerpt from Swift on page 24, distribute that page and ask the students to follow along as you read the excerpt. Then ask questions such as the following:

- Are there any mathematical errors in this excerpt?
- Could a slice of meat be an equilateral triangle? Can you draw it? How thick would it be? If you drew it to show its thickness, what figure would you draw?
- What word should Swift have used to describe this shape?

Initiating Activity
20 minutes

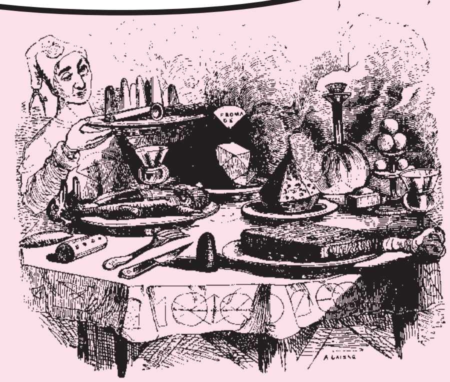
Have students work individually on Exercises ❶ and ❷ on page 25. When students have finished, write on the overhead or blackboard the headings, "2-D shapes" and "3-D figures." Invite students to come to the board or overhead one at a time to write an entry in either column and to justify their entry. Have the class discuss the entries in both lists. Ask the class if anyone knows what a cycloid is. If not, draw a sketch like this  and explain that a cycloid is an open curve. Ask students whether this is a 2-D shape or 3-D figure. The discussion should lead to the conclusion that it is neither because the boundary of a 2-D shape is a closed curve. Review orally the meanings of each of the terms in Exercise ❷.

Paired Activity
25 minutes

Before proceeding, ask for a show of hands to determine how many students know how to use the Pythagorean relationship to calculate distances on a grid. Also ascertain how many students know the formulas for the area of a triangle and parallelogram. Assign the appropriate template(s) (pp. 15–18) to students who need to learn or practise these concepts. (The answers are given on p. 93.) Group the remaining students in pairs. Distribute two sheets of centimetre paper to each pair (see p. 56), and have them work on Exercises ❸, ❹, and ❺.

ACTIVITY 1 – STUDENT PAGE

❶ In his description of the dinner, Gulliver confused some 2-dimensional shapes with 3-dimensional figures. Make a list of the 2-dimensional shapes he named and another list of the 3-dimensional figures. Rewrite Gulliver's first paragraph using the appropriate terms.

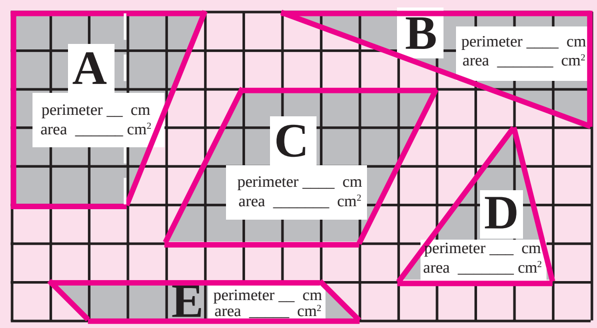


Gulliver's Dinner in Laputa, J.J. Grandville 1835

❷ Write a sentence and draw a sketch to explain the meaning of each term.

a) parallelogram
b) trapezoid
c) equilateral triangle
d) rhombus
e) rectangular prism
f) triangular prism

❸ Name these 2-dimensional shapes drawn on the centimetre grid below. Count squares to estimate the perimeter and area of each. Record your estimates.



❹ Write as many of these area formulas as you know.

a) The area of a rectangle given its length *l* and width *w*.
b) The area of a triangle given its height *h* and the length *b* of its base.
c) The area of a parallelogram given the length *l* of one side and the perpendicular distance *d* from it to the other parallel side.

Use the formulas you know to check your estimates in Exercise ❸.

❺. Draw each of these 2-dimensional shapes on centimetre paper.

a) a rectangle of area 20 cm² and perimeter 18 cm.
b) a parallelogram of area 24 cm² and perimeter 22 cm.
c) a quadrilateral of area 20 cm² and perimeter 20 cm.

Closure

In concluding this lesson, ask pairs to explain how they calculated the areas of the various shapes. Students should explain how they decomposed larger shapes into simple shapes such as right triangles. Others may explain how a right triangle is merely half of a rectangle. Show students how to calculate the perimeters using the Pythagorean relationship. Have students demonstrate on the blackboard how they applied the area formulas. When discussing student answers to Exercise ❺, encourage all possible answers and ask students whether they think there is more than one answer.

GULLIVER DINES WITH THE MATHEMATICIANS

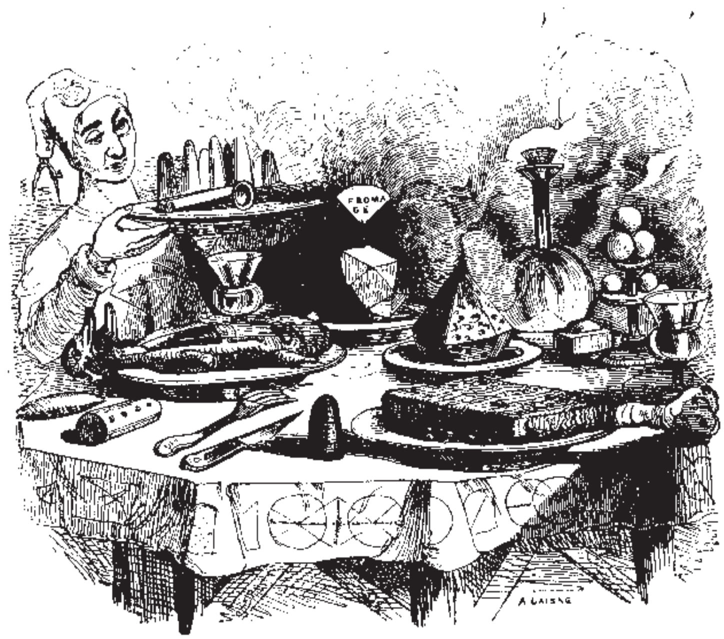
Gulliver's Travels is a popular tale of a traveller named Gulliver who sailed the oceans to strange and distant lands. Most people know of his visit to Lilliput, the land of the little people. Some know of his visit to Brobdingnag, island of the giants. But few have read the chapter about Gulliver's visit to Laputa – the land of the mathematicians. Some small excerpts from that visit are presented here in a slightly modified form, to modernize the old English in which this manuscript was written almost three centuries ago!

We had two courses of three dishes each. In the first course, there was a shoulder of mutton, cut into an equilateral triangle; a piece of beef into a rhombus and a pudding into a cycloid … The servants cut our bread into cones, cylinders, parallelograms and several other mathematical figures.

…Their ideas are perpetually expressed in lines and figures. To praise the beauty of an animal, they describe it in terms of rhombuses, circles, parallelograms, ellipses and other geometric terms.

Parts Unknown

❶ In his description of the dinner, Gulliver confused some 2-dimensional shapes with 3-dimensional figures. Make a list of the 2-dimensional shapes he named and another list of the 3-dimensional figures. Then rewrite Gulliver's first paragraph using the appropriate terms.

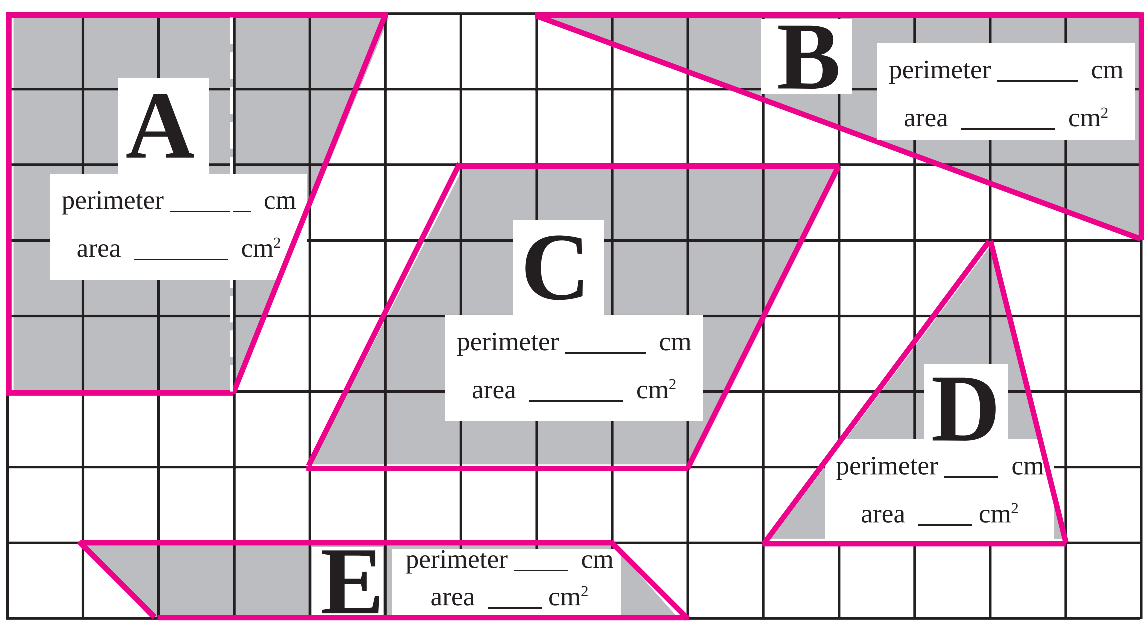


Gulliver's Dinner in Laputa, J.J. Grandville 1835

❷ Write a sentence and draw a sketch to explain the meaning of each term.

a) parallelogram b) trapezoid
c) equilateral triangle d) rhombus
e) rectangular prism f) triangular prism

❸ Name the 2-dimensional shapes drawn on the centimetre grid below.
Count squares to estimate the perimeter and area of each. Record your estimates.



❹ Write as many of these area formulas as you know.

a) The area of a rectangle given its length l and width w.
b) The area of a triangle given its height h and the length b of its base.
c) The area of a parallelogram given the length l of one side and the perpendicular distance d from it to the other parallel side.

Use the formulas you know to check your estimates in Exercise ❸.

❺ Draw each of these 2-dimensional shapes on centimetre paper.

a) a rectangle of area 20 cm² and perimeter 18 cm.
b) a parallelogram of area 24 cm² and perimeter 22 cm.
c) a quadrilateral of area 20 cm² and perimeter 20 cm.

Grade 7

Answer Key for Activity 1

❶

Two-dimensional Shapes	Three-dimensional Figures
• equilateral triangle • rhombus • parallelograms	• cones • cylinders

Since all solids are three-dimensional, we must replace the names of the two-dimensional shapes with corresponding three-dimensional solids. The revised paragraph with the appropriate terms may read something like this.

We had two courses of three dishes each. In the first course, there was a shoulder of mutton, cut into a prism having an equilateral triangle as a base; a piece of beef into a rhombohedron and a pudding into a cone. The servants cut our bread into cones, cylinders, rhombohedra and several other mathematical figures.

It is not expected that students will know the word *rhombohedron*, but the intent here is to help them understand the distinction between two-dimensional shapes and three-dimensional figures. It is also intended that the students have some fun with this exercise by improvising with descriptive words or explanations to describe the three-dimensional counterparts of simple two-dimensional shapes.

Note: The cycloid is a curve and is therefore one-dimensional, so it is even less appropriate than the other terms in describing a solid. We merely substituted the word *cone* in our revised paragraph.

❷ These definitions are not unique and should be understood but not memorized.

a) A *parallelogram* is a quadrilateral with opposite sides parallel.

b) A *trapezoid* is a quadrilateral with exactly one pair of parallel sides.

c) An *equilateral triangle* is a triangle with all sides of equal length.

d) A *rhombus* is a parallelogram with all sides of equal length.

e) A *rectangular prism* is a polyhedron with two parallel and congruent rectangular faces joined by four faces that are parallelograms.

f) A *triangular prism* is a polyhedron with two parallel and congruent triangular faces joined by three faces that are parallelograms.

❸ To estimate perimeter, students might use the fact that any diagonal line segment cutting across a centimetre square has a length greater than 0 cm and at most $\sqrt{2}$ cm. Using this approach, students might estimate the perimeter of A to be $13 + 5\sqrt{2} \approx 20$ cm. To estimate the area of A, students might count 17 full squares and 5 partial squares inside A and estimate the area to be 17 + 5(1/2) or 19.5 cm^2. The estimates can be further sharpened by assigning an area of 1 cm^2 to squares with more than half their areas inside A and an area of 0 to squares with less than half their areas inside A. Similarly, diagonal line segments that appear to be small can be assigned a length of 0 and larger segments, a length of $\sqrt{2}$. A reasonable range for each estimate of perimeter and area is given in the table.

SHAPE	Perimeter P in cm	Area A in cm^2
A	$18 \le P \le 20$	$19 \le A \le 21$
B	$18 \le P \le 23$	$11 \le A \le 13$
C	$18 \le P \le 21$	$19 \le A \le 21$
D	$9 \le P \le 15$	$6 \le A \le 10$
E	$16 \le P \le 17$	$6 \le A \le 8$

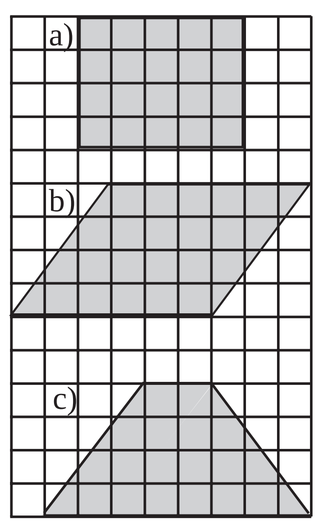
The classifications of the shapes A, B, C, D, and E and their calculated perimeters and areas are shown below.

A trapezoid	Perimeter 18.4 cm	Area 20 cm^2
B triangle	Perimeter 19.5 cm	Area 12 cm^2
C parallelogram	Perimeter 18.9 cm	Area 20 cm^2
D triangle	Perimeter 13.1 cm	Area 8 cm^2
E parallelogram	Perimeter 16.8 cm	Area 7 cm^2

❹ a) $A = l \times w$
b) $A = h \times b/2$
c) $A = l \times d$

It is important for students to check the estimates they obtained by counting squares, so they can refine their estimation techniques. This method will be used in the Grade 8 unit to approximate the area of a circle.

❺ Three shapes that satisfy the given conditions are shown on the grid.



The scoring guide presented below has been developed using student responses on a field test conducted in 1998. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 1

	Level 1	Level 2	Level 3	Level 4
CONCEPTS **Creating Definitions of 2-D Shapes & 3-D Figures** (exercise ❷) M 7-1	• Several definitions are incomplete and/or contain some significant errors.	• Most definitions are appropriate and diagrams correct. • Some definitions contain incomplete or inaccurate descriptions.	• Definitions are mostly complete and usually include accurate descriptions. • Diagrams are correct.	• Complete and accurate definitions are given.
APPLICATION **Estimation & Calculation of Perimeter & Area** (exercises ❸ and ❹) M 7-5, M 7-6, M 7-8	• Formulas are incorrect and improperly applied.	• Formulas are correct but applied improperly.	• Formulas are correct but minor errors are evident in their application.	• Formulas are correct and correctly applied.

Achievement Levels Defined by the Ministry of Education and Training

Level 1 Identifies achievement that falls much below the provincial standard.

Level 2 Identifies achievement that approaches the standard.

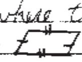
Level 3 The "provincial standard," identifies a high level of achievement of provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.

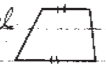
Level 4 Identifies achievement that surpasses the standard.

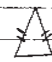
WHAT YOU MIGHT SEE

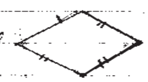
Understanding of Concepts: Definitions of 2-D Shapes & 3-D Figures

Level 2

3) a) parallelogram - a 2-dimensional figure where there are two parallel sides opposite to each other 

b) trapezoid - a 2-dimensional figure with one parallel side the other two are not parallel 

c) equilaterel - a triangle with two sides the same length. 

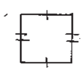
d) rhombus - a parallelogram but all the sides are the same length 

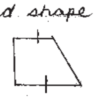
e) rectangular prism - is a 3-dimensional figure with parallel edges. 

f) triangular prism - is a 3 dimensional figure with two triangle faces parallel to each other. 

This student has drawn all the shapes and figures correctly, except for the equilateral triangle which has been confused with an isosceles triangle. The written definitions of the prisms do not define these figures uniquely and are therefore incomplete. The written definitions of the parallelogram and trapezoid neglect to mention the number of sides. The student knows what these shapes look like but needs some coaching in expressing these ideas.

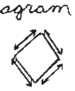
Level 3

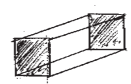
2 a) Parallelogram - a four sided figure with both pairs of opposite sides equal in length and distance from eachother. 

b) Trapezoid - a four sided shape with exactly one pair of parallel sides. 

c) Equilateral Triangle - a triangle having all sides with an equal angle. 

60°

d) rhombus - a parallelagram with four equal sides in length. 

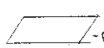
e) Rectangular Prism - a geometric solid of two square faces, and four rectangular sides which join the faces. 

f) Triangular Prism - a geometric solid having two rectangular faces, and three rectangular sides which join the faces. 

The definitions of the 2-D shapes are correct and complete. The definitions of the prisms have two errors that are underlined. The diagrams suggest that the student knows the meanings of the terms, but may need to be reminded that a rectangular prism need not have square ends.

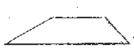
Level 4

2) a) A parallelogram is a four-sided figure with opposite sides parallel.



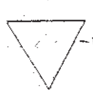
-Parallelogram-

b) A trapezoid is a quadrilateral with exactly one pair of parallel lines.



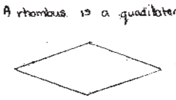
-Trapezoid-

c) An equilateral triangle is a 3-sided figure with all 3 sides being of equal length.



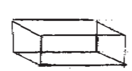
-Equilateral Triangle

d) A rhombus is a quadilateral with four sides of equal length.



-Rhombus-

e) A rectangular prism is a prism whose top and bottom surfaces are congruent rectangles.



-Rectangular Prism-

f) A triangular prism is a five-sided figure with two ~~ends~~ being congruent triangles and the remaining three sides being rectangles.

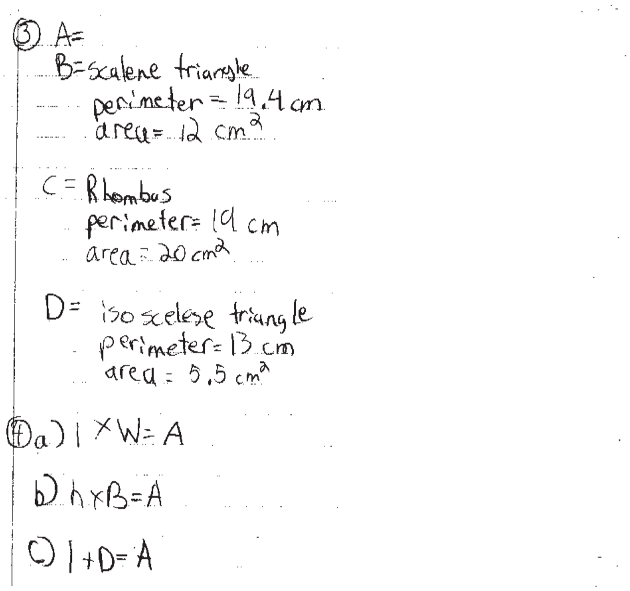
-Triangular Prism-

These definitions are complete and accurate for this grade level. The student may need to be reminded that the "sides" of 3-D figures are referred to as "faces." The definition of a triangular prism given here is actually the definition of a *right* triangular prism. However the more general definition of prisms (see Answer Key p. 26) should be mentioned but not stressed at this grade level.

What You Might See

Application of Mathematical Procedures: Estimation & Calculation of Perimeter & Area

Level 1



3) A=
B= scalene triangle
perimeter = 19.4 cm
area = 12 cm^2

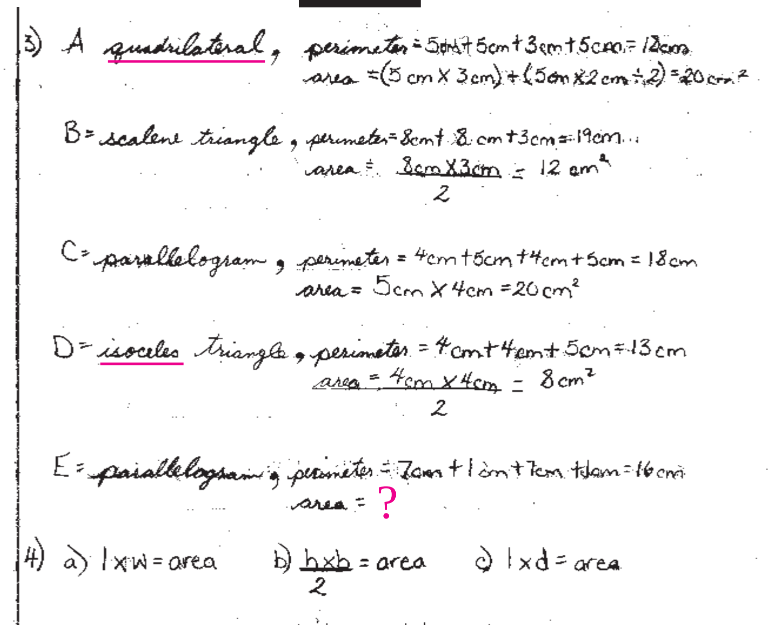
C= Rhombus
perimeter = 19 cm
area = 20 cm^2

D= isoscelese triangle
perimeter = 13 cm
area = 5.5 cm^2

4) a) l × W = A
b) h × B = A
c) l + D = A

Of the three area formulas given, only the formula for the area of a rectangle is correct. Estimates of the area and perimeter are given for only three of the five shapes. The perimeters fall within the range of normal error and the areas given for the right triangle and parallelogram are correct. The area of triangle D is incorrect.

Level 3



3) A quadrilateral, perimeter = 5cm + 5cm + 3cm + 5cm = 18cm
area = (5 cm × 3 cm) + (5cm × 2 cm ÷ 2) = 20 cm^2

B = scalene triangle, perimeter = 8cm + 8 cm + 3cm = 19cm
area = $\frac{8cm \times 3cm}{2}$ = 12 cm^2

C = parallelogram, perimeter = 4cm + 5cm + 4cm + 5cm = 18cm
area = 5cm × 4cm = 20cm^2

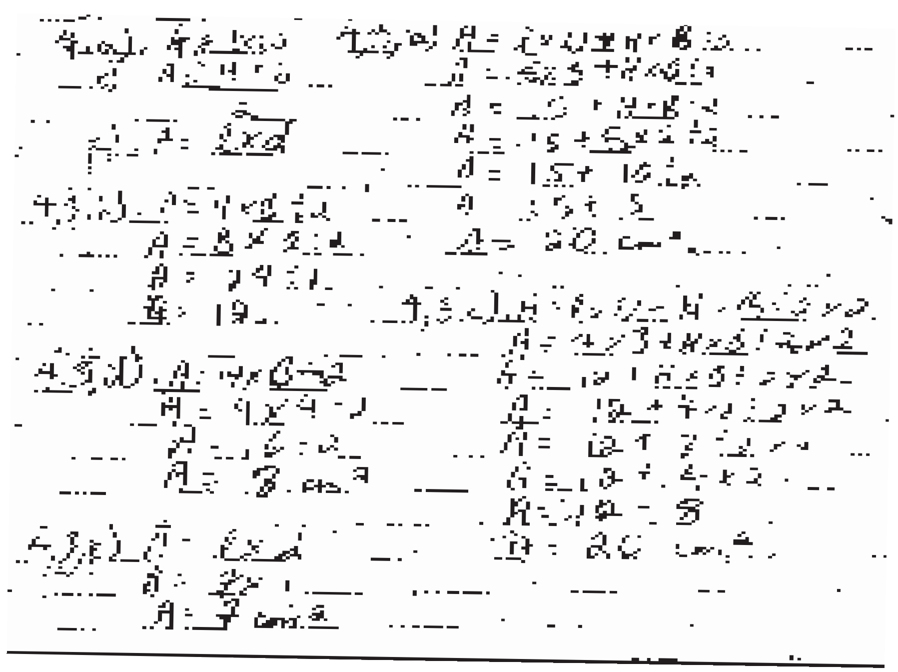
D = isoceles triangle, perimeter = 4cm + 4cm + 5cm = 13cm
area = $\frac{4cm \times 4cm}{2}$ = 8cm^2

E = parallelogram, perimeter = 7cm + 1cm + 7cm + 1cm = 16cm
area = ?

4) a) l × w = area b) $\frac{h \times b}{2}$ = area c) l × d = area

All three formulas given for area are correct. The perimeters given were within the limits of acceptable error and the areas of shapes A through D were correct. The student did not estimate or calculate the area of parallelogram E although the appropriate formula was known. It is not clear whether this was merely an oversight.

Level 4



All three formulas given for area are correct. The perimeters (not shown here) given were within the limits of acceptable error.

The calculations show that the student understands how to apply the formulas. In calculating the area of trapezoid A, the student divides it into a rectangle and a triangle and applies both area formulas. To calculate the area of parallelogram C, the student breaks it into a rectangle and two congruent triangles and then applies the appropriate formulas.

Activity 2 – Teacher Edition

The Mathematicians Transform Rectangles into Trapezoids

Expectations Addressed

M 7-1 create definitions of measurement concepts.

M 7-2 describe measurement concepts using appropriate measurement vocabulary.

M 7-4 make increasingly more informed and accurate measurement estimations based on an understanding of formulas and the results of investigations.

M 7-5 understand that irregular two-dimensional shapes can be decomposed into simple two-dimensional shapes to find the area and perimeter.

M 7-6 estimate and calculate the perimeter and area of an irregular two-dimensional shape (e.g., trapezoid, hexagon).

M 7-7 develop the formula for finding the area of a trapezoid.

M 7-8 estimate and calculate the area of a trapezoid, using a formula.

M 7-9 draw a trapezoid given its area and/or perimeter.

Context

In this activity, we set the stage for the human drama that motivates the calculation of area. Tension between the theoretical mathematicians and the king's tax appraiser is triggered by the king's imposition of a special tax on lots containing more than two right angles. The mathematicians respond by changing the shapes of their lots from rectangles to right-angled trapezoids, in such a way that the areas are unchanged. In this way they reduce the number of right angles in each lot to two, thereby avoiding the special tax. The king will respond to this tax avoidance scheme in Activity 3. (Stay tuned.)

Activity 2 – Student Page

The Mathematicians Transform Rectangles into Trapezoids

Gulliver observed, with some contempt, that the mathematicians seemed to avoid practical matters. They built their homes without right angles and located their houses on odd-shaped lots.

These mathematicians are under continual stress, never enjoying a minute's peace of mind, for they are always working on some problem that is of no interest or use to the rest of us.

¤

…Their houses are very ill built, the walls bevil, without one right angle in any apartment; and this defect ariseth from the contempt they bear for practical geometry. They despise it as vulgar and impure… Although they can use mathematical tools like ruler, compasses, pencil, and paper, they are clumsy and awkward in the common actions and behaviours of life.

Gulliver was apparently unaware of the reasons why the mathematicians constructed their buildings (and their lots) in unsymmetrical shapes. Legend tells how the king, in his attempt to raise more revenue from his people, levied a special tax on any lot that contained more than two right angles. Two mathematicians, Alpha and Beta, with adjoining rectangular lots, reshaped their lots as shown, to avoid this special tax.

Before the Special Tax

After the Special Tax

Exercise ❶ is intended to ensure that students understand the meaning of the term *trapezoid* and realize that a trapezoid can have two or four (but not three) right angles. This enables them to understand how the transformation of the lots from rectangular to trapezoidal shape avoids the special tax. Exercise ❷ has students measure the dimensions of Alpha's and Beta's rectangular lots and calculate their areas. Then students trace and cut out the lots after they have been reshaped. By placing the cut outs on centimetre paper, the students can verify that the areas of the lots have not been changed. Furthermore they discover that multiplying the sum of the lengths of the parallel sides of a (right-angled) trapezoid by the distance between them yields double the area. This leads them to an informal discovery of the formula for the area of a (right-angled) trapezoid. In Exercises ❸ and ❹, students investigate how to transform a given rectangle into a trapezoid of equal area with and without centimetre paper.

The Lesson Launch

10 minutes

Investigating and applying a formula for the area of a trapezoid may be intrinsically interesting to some students, but others (perhaps many) would regard the experience as "boring." For this reason, we have attempted to embed the exploration in a context involving the drama of matching wits – the king and his tax appraiser vs. the theoretical mathematicians.

By tapping into the rich fantasy of *Gulliver's Travels*, you can integrate some language arts into the lesson and, in the process, motivate some mathematical learning. To achieve this, it is important that students know something about *Gulliver's Travels.* One way to provide such background information is to ask anyone who knows the story to describe it to the class. Then distribute page 32 and invite someone to read the excerpt on the scroll. Have the students read for themselves the paragraph below the scroll. Then pose questions such as the following to launch the lesson:

- *Why did the mathematicians reshape their lots?*
- *What were the shapes of the lots before and after the tax?*
- *How many right angles did each lot have?*
- *Why did the mathematicians want to keep the areas of their lots unchanged?*
- *Do you think the mathematicians were justified in changing the shapes of their lots? Explain why or why not.*

Paired Activity

20 minutes

Group students in pairs to complete Exercises ❶ and ❷ in their notebooks. Distribute two copies of page 33, two sheets of centimetre paper (see p. 56), and scissors to each pair. When most pairs are finished, ask students to explain any relationship they found between the length of a lot before the tax and the sum of the lengths of the parallel sides after tax. Prompt them to explain how to use this relationship to calculate the area of a trapezoid (containing a right angle).

Individual Activity

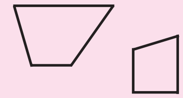
20 minutes

Assign Exercises ❸ and ❹ for students to do individually. Many students will discover that a line segment drawn through the midpoint of the boundary between A and B divides it into two trapezoids with the same areas as rectangles A and B. Help students who experience difficulty by suggesting that they fold their rectangle in half along a line parallel to its length. The point where the fold intersects the boundary of A and B is the point through which any line segment can be drawn to yield the desired result.

Activity 2 – Student Page

The Mathematicians Transform Rectangles into Trapezoids

By reconstructing their lots as shown on page 32, the mathematicians changed each rectangular lot into a *trapezoid*.



❶ a) The diagram on the right shows two trapezoids. Write a sentence to define a trapezoid. Check your definition with a dictionary.

b) How many right angles has each trapezoid shown here? Do all trapezoids have the same number of right angles? Explain.

c) Did Alpha and Beta have to pay the special tax on their new lots? Explain.

❷ a) Measure the length and width in millimetres of Alpha's and Beta's lots **before** the special tax was imposed. Record in the table on the left below.

Before Tax			
	Length	Width	Area
Alpha			
Beta			

After Tax			
	Sum of the Lengths of the Parallel sides	Distance between the Parallel sides	Area
Alpha			
Beta			

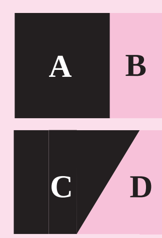
b) Trace and cut out both lots as they were after the special tax. Place your cut outs on centimetre paper to determine the area of each lot and the lengths of the parallel sides. Record in the table on the right.

c) Did Alpha and Beta change the areas of the lots when they reshaped them? Explain.

d) Compare the length of Alpha's rectangular lot to the sum of the lengths of the parallel sides of Alpha's trapezoidal lot. Repeat for Beta's lot. Describe what you discover.

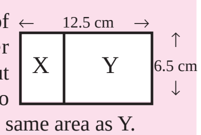
e) Explain how to calculate the area of a trapezoid containing a right angle, given the lengths of its parallel sides and distance between them.

❸ a) Draw two rectangles of length 9 cm and width 6 cm on centimetre paper. Divide one of the rectangles into two rectangles A and B with dimensions 5 cm × 6 cm and 4 cm × 6 cm.



b) Use what you learned in Exercise ❷ to divide the other rectangle into trapezoids C and D so the areas of A and C are the same and the areas of B and D are the same. Explain how you did this. How many ways do you think this can be done?

❹ a) Draw a 12.5 cm × 6.5 cm rectangle on a sheet of paper. Divide your rectangle into two other rectangles X and Y and record their areas. Cut out your rectangle and divide it into two trapezoids so that one has the same area as X and the other the same area as Y.



b) Measure the dimensions of each trapezoid and calculate its area as in ❷ b. Record the areas of the trapezoids and verify that they are equal to the areas of X and Y.

Closure

To consolidate the students' understanding of a procedure for transforming a rectangle into a trapezoid of the same area, invite students to describe their findings in ❸ and ❹. Ensure that they understand there are (infinitely) many ways to do this. Guide them to deduce a formula for the area of a trapezoid given the lengths of its parallel sides and the distance between them.

The Mathematicians Transform Rectangles into Trapezoids

Gulliver observed, with some contempt, that the mathematicians seemed to avoid practical matters. They built their homes without right angles and located their houses on odd-shaped lots.

These mathematicians are under continual stress, never enjoying a minute's peace of mind, for they are always working on some problem that is of no interest or use to the rest of us.

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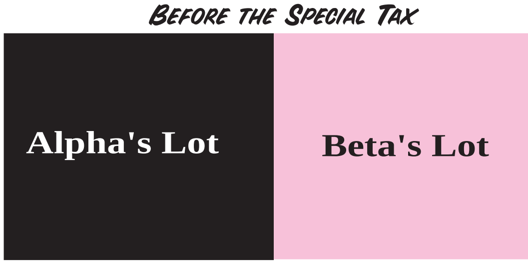
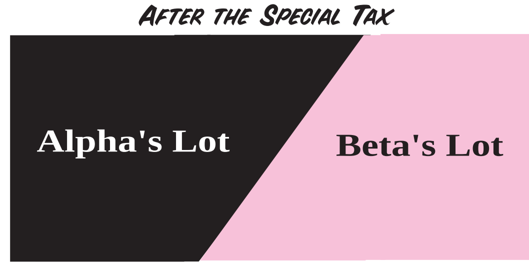
…Their houses are very ill built, the walls bevil, without one right angle in any apartment; and this defect ariseth from the contempt they bear for practical geometry. They despise it as vulgar and impure…Although they can use mathematical tools like ruler, compasses, pencil, and paper, they are clumsy and awkward in the common actions and behaviours of life.

Gulliver was apparently unaware of the reasons why the mathematicians constructed their buildings (and their lots) in unsymmetrical shapes. Legend tells how the king, in his attempt to raise more revenue from his people, levied a special tax on any lot that contained more than two right angles. Two mathematicians, Alpha and Beta, with adjoining rectangular lots, reshaped their lots as shown, to avoid this special tax.

Before the Special Tax

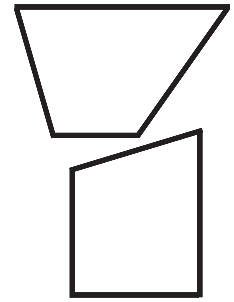


After the Special Tax



Activity 2 – Student Page

By reconstructing their lots as shown on page 32, the mathematicians Alpha and Beta changed each rectangular lot into a *trapezoid*.



❶ a) The diagram on the right shows two trapezoids. Write a sentence to define a trapezoid. Check your definition with a dictionary.

b) How many right angles has each trapezoid shown here? Do all trapezoids have the same number of right angles? Explain.

c) Did Alpha and Beta have to pay the special tax on their new lots? Explain.

❷ a) Measure the length and width in millimetres of Alpha's and Beta's lots **before** the special tax was imposed. Record in the table on the left.

Before Tax			
	Length	Width	Area
Alpha			
Beta			

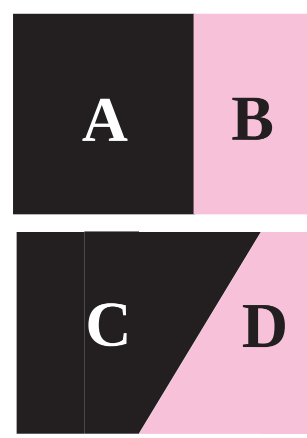
After Tax			
	Sum of the Lengths of the Parallel sides	Distance between the Parallel sides	Area
Alpha			
Beta			

b) Trace and cut out both lots as they were after the special tax. Place your cut outs on centimetre paper to determine the area of each lot and the lengths of the parallel sides. Record in the table on the right.

c) Did Alpha and Beta change the areas of the lots when they reshaped them? Explain.

d) Compare the length of Alpha's rectangular lot to the sum of the lengths of the parallel sides of Alpha's trapezoidal lot. Repeat for Beta's lot. Describe what you discover.

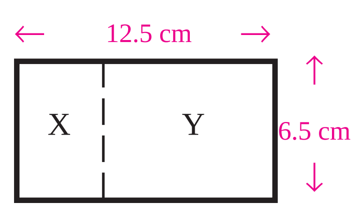
e) Explain how to calculate the area of a trapezoid containing a right angle, given the lengths of its parallel sides and distance between them.



❸ a) Draw two rectangles of length 9 cm and width 6 cm on centimetre paper. Divide one of the rectangles into two rectangles A and B with dimensions 5 cm × 6 cm and 4 cm × 6 cm.

b) Use what you learned in Exercise ❷ to divide the other rectangle into trapezoids C and D so the areas of A and C are the same and the areas of B and D are the same. Explain how you did this. How many ways do you think this can be done?

❹ a) Draw a 12.5 cm × 6.5 cm rectangle on a sheet of paper. Divide your rectangle into two other rectangles X and Y and record their areas. Cut out your rectangle and divide it into two trapezoids so that one has the same area as X and the other the same area as Y.



b) Measure the dimensions of each trapezoid and calculate its area as in ❷ b. Record the areas of the trapezoids and verify that they are equal to the areas of X and Y.

Grade 7
Answer Key for Activity 2

❶ a) A *trapezoid* is a quadrilateral with exactly one pair of parallel sides. (Sometimes a trapezoid is defined as a quadrilateral with *at least* one pair of sides parallel. This includes parallelograms as special trapezoids. However, the glossary on page 84 of The Ontario Curriculum, Grades 1–8 uses the more restrictive form given above.)

b) As seen in the diagrams, a trapezoid may have no right angles or two right angles. It cannot have exactly one right angle, for otherwise the other parallel side would also have a right angle. Also, it cannot have exactly three right angles because the remaining angle would have to have measure 90˚ and then it would have four right angles.

c) Alpha and Beta would not have to pay a tax on their new lots because the special tax was levied on the lots with more than two right angles. Trapezoids cannot have more than two right angles.

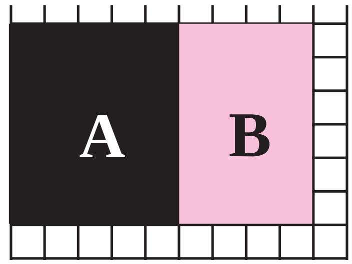
❷ a) The completed tables are shown here.

Before Tax			
	Length	**Width**	**Area**
Alpha	36 mm	30 mm	10.8 cm^2
Beta	34 mm	30 mm	10.2 cm^2

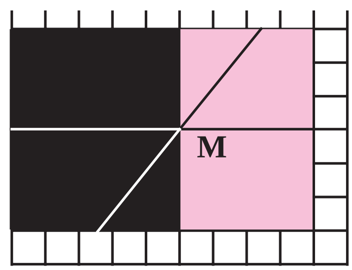
After Tax			
	Sum of the Lengths of the Parallel Sides	**Distance between the Parallel Sides**	**Area**
Alpha	47 + 25 = 72 mm	30 mm	10.8 cm^2
Delta	23 + 45 = 68 mm	30 mm	10.2 cm^2

b) The areas are shown in the tables above.
c) No. The areas of the lots remained unchanged.
d) The sum of the lengths of the parallel sides of Alpha's trapezoidal lot was double the length of Alpha's rectangular lot. This was also true for Beta's lots.
e) To find the area of a trapezoid, find the average length of its two parallel sides and multiply by the distance between them.

❸ a) The diagram below shows two rectangles A and B drawn drawn to scale on centimetre paper.

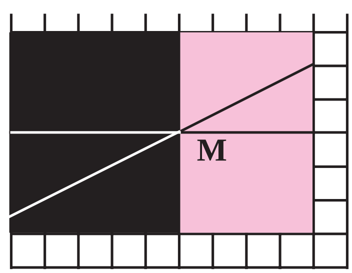


b) To divide the large rectangle above into trapezoids C and D so that the areas of A and C are equal and the areas of B and D are equal, we draw a line through the midpoint M of the boundary between A and B. (We can locate the midpoint by folding the rectangle perpendicular to the boundary.) Then we draw any line segment through M, and it divides the rectangle into trapezoids C and D.



Since any line through M produces a different pair of trapezoids, there are an infinite number of ways of dividing the rectangle into trapezoids so that the areas are preserved.

CHALLENGE Ask students whether the areas remain unchanged if the line segment through M is allowed to move past the corners as in the diagram below.



You can verify that rotating past the corners does not preserve areas unless the two rectangles are of equal area.

The scoring guide presented below has been developed using student responses on a field test conducted in 1998. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

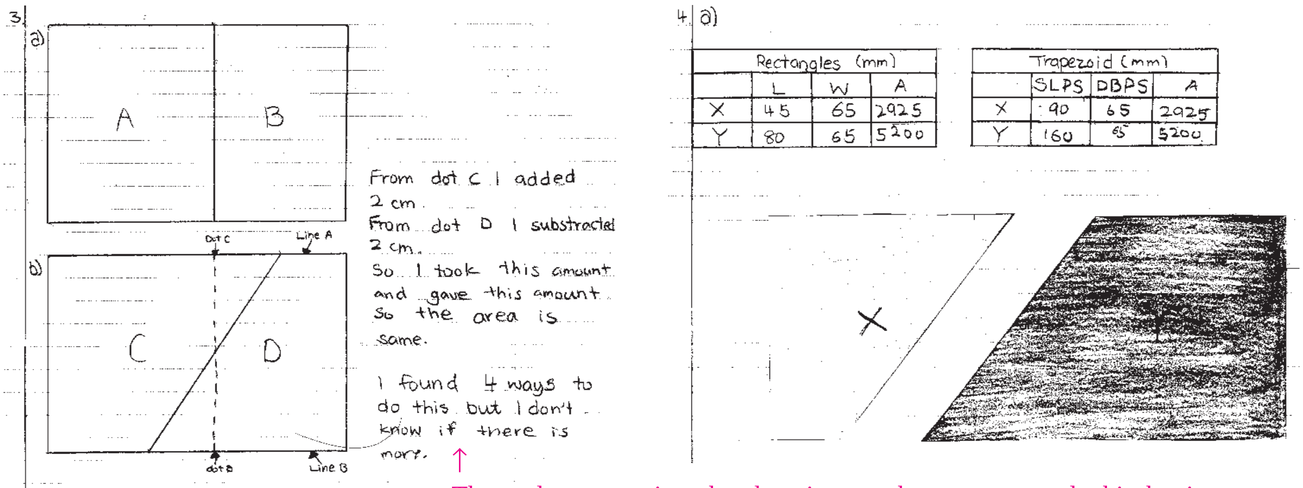
Scoring Guide for Activity 2

	Level 1	Level 2	Level 3	Level 4
PROBLEM SOLVING **Selection of an Appropriate Strategy for Constructing a Trapezoid with The Same Area as a Given Rectangle** (exercises ❸ & ❹) M 7-6, M 7-7, M 7-8, M 7-9	• unable to describe an appropriate strategy for changing a rectangle into a right trapezoid of the same area.	• describes an appropriate strategy for changing a rectangle into a right trapezoid of the same area. • applies the strategy with minor errors to a given rectangle.	• describes an appropriate strategy for changing a rectangle into a right trapezoid of the same area. • applies the strategy correctly to a given rectangle.	In addition to Level 3: • indicates that there are an infinite number of ways of changing a rectangle into a right trapezoid of the same area.
CONCEPTS **Understands How To Determine the Area of a Right Trapezoid** (exercise ❷) M 7-4, M 7-6, M 7-7, M 7-8, M 7-9.	• The response to Exercise ❷ includes fewer than two of the following elements: • completed tables with measurements correct to within 2 mm and no computational errors. • statement that the areas of the lots did not change. • discovery that the sum of the lengths of the parallel sides after tax is double the length before tax. • appropriate procedure or formula is given for calculating the area of a right trapezoid.	• The response to Exercise ❷ includes two of the following elements: • completed tables with measurements correct to within 2 mm and no computational errors. • statement that the areas of the lots did not change. • discovery that the sum of the lengths of the parallel sides after tax is double the length before tax. • appropriate procedure or formula is given for calculating the area of a right trapezoid.	• The response to Exercise ❷ includes three of the following elements: • completed tables with measurements correct to within 2 mm and no computational errors. • statement that the areas of the lots did not change. • discovery that the sum of the lengths of the parallel sides after tax is double the length before tax. • appropriate procedure or formula is given for calculating the area of a right trapezoid.	• The response to Exercise ❷ includes all of the following elements: • completed tables with measurements correct to within 2 mm and no computational errors. • statement that the areas of the lots did not change. • discovery that the sum of the lengths of the parallel sides after tax is double the length before tax. • appropriate procedure or formula is given for calculating the area of a right trapezoid.

WHAT YOU MIGHT SEE

Problem Solving: Selecting a Strategy for Constructing a Trapezoid of Given Area

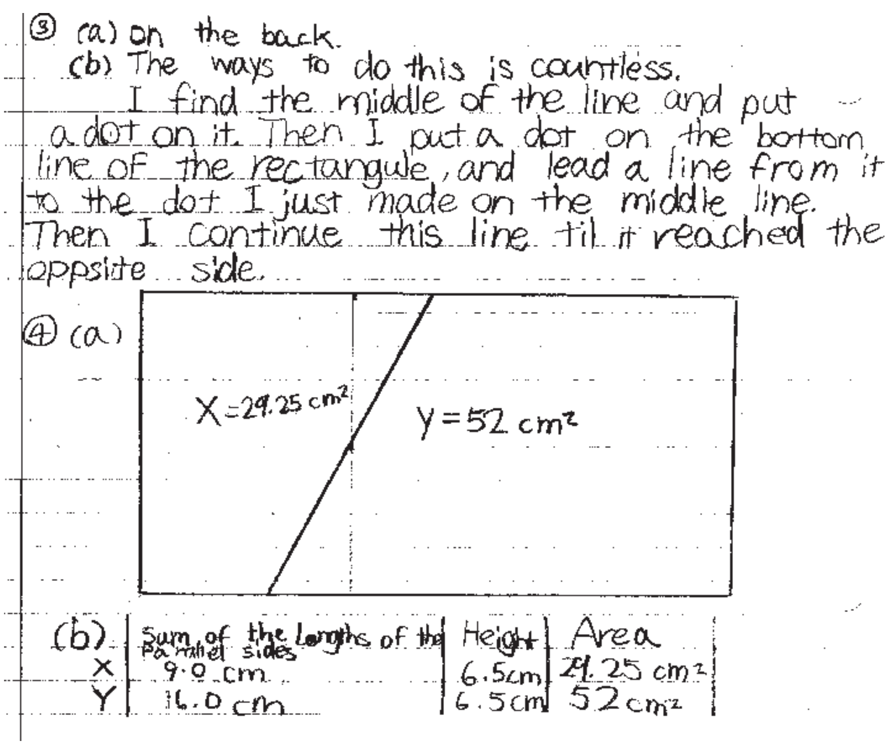
Level 3



The student recognizes that there is more than one way to do this, but is unaware that continuous turning of the boundary line yields an infinite number of solutions.

The student has described very clearly an appropriate strategy for partitioning the rectangle into two right trapezoids such that the areas are preserved. The strategy was also applied correctly. She used centimetre paper in her rough work and was thinking about solutions for which the dimensions of the trapezoids are integers. This led her to think that there were only four solutions.

Level 4



The student has recognized that there are an infinite number of different trapezoids that can be constructed so that areas are preserved. The student whose work is shown above moved 2 cm right along the top edge of the rectangle and then 2 cm left along the bottom edge and then joined the two points. This student finds the midpoint of the boundary between the two rectangles and then draws any line segment through that point. Each such line segment is a solution to the problem. This approach is a little more sophisticated than the strategy above and it reveals an interesting property of all partitions satisfying the conditions of the problem. That is, all the line segments pass through the mid point of the boundary.

WHAT YOU MIGHT SEE

CONCEPTS: UNDERSTANDS HOW TO DETERMINE THE AREA OF A RIGHT TRAPEZOID

Level 3

Before Tax			
	Length	Width	Area
Alpha	3cm	3.5cm	10.5cm²
Beta	3cm	3.5cm	10.5cm²

After Tax			
	Sum of the Lengths of the Parallel sides	Distance between the Parallel sides	Area
Alpha	6.5cm	3cm	10.5cm²
Beta	6.5cm	3cm	10.2cm²

↑ table values outside the 2 mm range

② (a) Done
(b) Done(cut out isn't the fastest way)
(c) No. because the area of the land they lost equal the land they newly got.  land As ← recognition that the areas of the lots did not change
(d) the lenghth are the same
(e) Two parallel side plus together x height ÷ 2 (land / middle / last) ← appropriate procedure

The lengths of the sides for lot Alpha given in the tables are outside the 2 mm tolerance (36 ± 2 mm and 72 ± 2 mm). This may have been a consequence of confusing the terms "length" and "width." Also the area given in the Beta row in the After Tax table does not match the product of the other dimensions. All the other elements listed in the Level 3 column of the Scoring Guide on page 35 are present.

Level 4

2 a)

Before Tax (mm)			
	L	W	A
Alpha	36	30	1080
Beta	33	30	990

b)

After tax (mm)			
	SLPS	DBPS	A
Alpha	72	30	1080
Beta	66	30	990

c) No they didn't because if you lengthen such much on top of the lot and shorten such much on the bottom they get a different shape but same area.

d) The sum of the lengths of the parallel sides of Alpha and Beta's trapezoids are equle to the doubled length of each rectangular lot.

e) 1. Find the sum of the lengths of the parallel sides.

2. Multiply the sum by the distance between the parallel sides.

3. You divide the answer in 2.

All the elements in the Level 4 column of the Scoring Guide on page 35 are present.

ACTIVITY 3 – TEACHER EDITION

THE KING MOVES FROM ANGLES TO AREA

Expectations Addressed

M 7-4 make increasingly more informed and accurate measurement estimations based on an understanding of formulas and the results of investigations.

M 7-5 understand that irregular two-dimensional shapes can be decomposed into simple two-dimensional shapes to find the area and perimeter.

M 7-6 estimate and calculate the perimeter and area of an irregular two-dimensional shape (e.g., trapezoid, hexagon).

M 7-7 develop the formula for finding the area of a trapezoid.

M 7-8 estimate and calculate the area of a trapezoid, using a formula.

M 7-9 draw a trapezoid given its area and/or perimeter.

Context

This activity escalates by a magnitude the battle of wits between the tax appraiser and the mathematicians. Reacting to the mathematicians' reshaping of their lots into trapezoids, the king's tax appraiser announces that taxes will be levied on lots according to their areas and not the number of right angles they contain.

To add insult to injury, the tax appraiser explains his rule for calculating the area of a trapezoid. At this point students have discovered this rule as it applies to trapezoids containing a right angle (see Activity 2, pp. 32–33) but they have not yet established a formula for the area of a trapezoid containing no right angles.

Exercise ❶ of this activity has students write a formula to describe the tax appraiser's rule for calculating the area of a trapezoid. It invites students to conjecture whether this formula applies to trapezoids that do not contain a right angle. In Exercise ❷, the student is to reconcile the tax appraiser's formula with the formula developed in Activity 2. In ❷b, the student applies the tax appraiser's formula to compute the areas of two trapezoids on a grid. Trapezoid B does not contain a right angle. Then in Exercise ❸, the students calculate the areas of the trapezoids A and B by decomposing them into rectangles and triangles. By comparing their areas with those obtained from the formula in Exercise ❷, the student discovers that the tax appraiser's formula yields the correct areas for these trapezoids. Exercise ❹ leads students to discover the formula for the area of a right trapezoid by fitting it together with a congruent copy to form a rectangle. This technique is repeated in ❹ d to lead students to discover the formula for the area of a general trapezoid.

ACTIVITY 3 – STUDENT PAGE

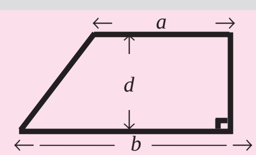
THE KING MOVES FROM ANGLES TO AREA

You may recall in Activity 2 that the king levied a special tax on lots with more than two right angles. In response, the mathematicians reshaped their rectangular lots into trapezoids of the same area. In this way they preserved the size of each lot and escaped the new tax. The king was not amused, and sent his tax appraiser to announce new tax measures.

The king is quite clearly annoyed,
For the taxes you tried to avoid.
By changing your lots
From rectangular plots,
To cleverly-shaped trapezoids.

So the king ordered me to advise
That he will tax each lot by its size;
For he doesn't care
Trapezoid or square,
The area only applies.

Your tax appraiser's no fool,
He calculates fast without tools.
Mean parallel side
Times measurement wide
Is his trapezoid area rule.



❶ a) How did the king revise the special tax provision so that taxes would not depend on the shape of the lot?

b) What does the tax appraiser mean by "mean parallel side"? by "measurement wide"?

c) Describe in your own words how the tax appraiser calculates the area of a trapezoid.

d) Write as a formula the tax appraiser's rule for calculating the area of a trapezoid that has parallel sides of length a and b if the distance between these sides is d units. Do you think this formula works for a trapezoid that has no right angles? Give a reason for your answer.

The Lesson Launch 10 minutes

Distribute copies of page 40 to all students. Remind them of the special tax that was levied on right angles and how the mathematicians reshaped their lots into trapezoids. Then read to the class, with drama and panache, the three limericks contained in the king's edict. Try to draw the students into the battle of wits by posing questions such as the following:

- *Has the king outwitted the mathematicians?*
- *What do you think the mathematicians will do now?*
- *What would you do if you were one of the mathematicians?*

Initiating Activity 10 minutes

Discuss Exercises ❶ a, b, and c with the class. Ensure that they understand the tax appraiser's rule. Then present Exercise ❶ d and ask students to work on it for a couple of minutes. They may work alone or with a partner. When most students have found a formula, invite a student who achieved the correct formula to come to the blackboard or overhead projector to explain the formula and show how it matches the tax appraiser's rule.

Individual Activity 15 minutes

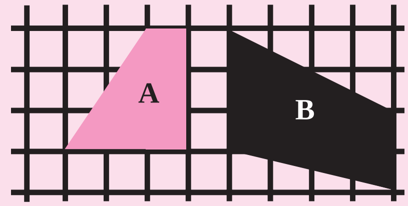
Distribute page 41 to all students. Have students work individually on Exercises ❷ and ❸. Circulate around the room to ensure that all students are correctly applying the formula for the area of a trapezoid. Check also that they are able to calculate the areas of the triangles by applying the appropriate formula.

Paired Activity 15 minutes

When students have completed Exercise ❸, review the formulas for the area of a triangle and the area of a parallelogram. Divide the students into pairs and provide each pair with a sheet of centimetre paper (see p. 56), tape, and a pair of scissors. Assign Exercise ❹ and circulate around the classroom, checking to see how many of the pairs are able to express the areas of the rectangle and parallelogram in terms of *a*, *b*, and *d*.

ACTIVITY 3 – STUDENT PAGE

❷ a) Is the tax appraiser's rule for calculating the area of a trapezoid the same as the formula you discovered in Activity 2? Explain your answer.

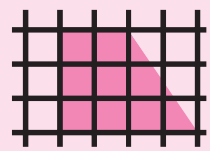


b) Use the tax appraiser's rule to calculate the areas of the trapezoids drawn on this centimetre grid.

❸ a) Draw a line segment to divide trapezoid A in Exercise ❷ into a right triangle and a rectangle. Calculate the areas of the rectangle and triangle to find the area of trapezoid A. Compare with your answer in ❷ b.

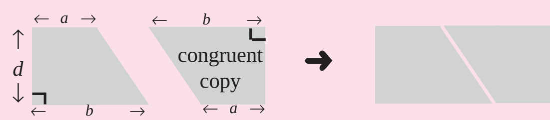
b) Divide trapezoid B in Exercise ❷ into two triangles. Then use the formula for the area of a triangle to calculate the area of trapezoid B. Compare with your answer in Exercise ❷ b.

❹ a) Draw a trapezoid like the one on the right on centimetre paper and count squares to determine its area. Draw another trapezoid congruent to it. Cut out both trapezoids and fit them together to form a rectangle.



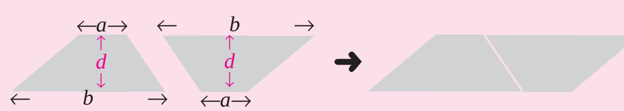
b) Record the area of the rectangle and the area of each trapezoid in ❹ a.

c) A congruent copy of the trapezoid below is made and they are fitted together to form a rectangle as shown.



Write an expression for the area of the rectangle and for the area of each trapezoid in terms of *a*, *b*, and *d*.

d) A congruent copy of the trapezoid below is made and they are fitted together to form a parallelogram as shown.



CHALLENGE Write an expression for the area of the parallelogram and for the area of the trapezoid in terms of *a*, *b*, and *d*. Show your work.

Closure

Invite a student who discovered the correct formula for the area of a right-angled trapezoid to display the formula on the blackboard and explain how it was derived. Then allow a few minutes for the students who were unable to obtain a formula for the area of a general trapezoid to try again. Once most students have discovered a correct formula for the area of a trapezoid, sketch a general trapezoid on the blackboard and invite a student to measure the lengths of the parallel sides and the distance between them and to calculate its area.

THE KING MOVES FROM ANGLES TO AREA

You may recall in Activity 2 that the king levied a special tax on lots with more than two right angles. In response, the mathematicians reshaped their rectangular lots into trapezoids of the same area. In this way they preserved the size of each lot and escaped the new tax. The king was not amused, and sent his tax appraiser to announce new tax measures.

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For the taxes you tried to avoid.
By changing your lots
From rectangular plots,
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So the king ordered me to advise
That he will tax each lot by its size;
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Trapezoid or square,
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He calculates fast without tools.
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Times measurement wide
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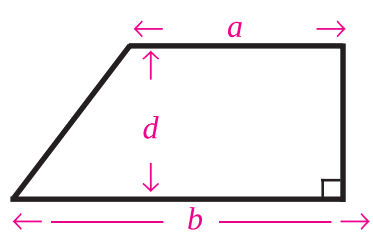


❶ a) How did the king revise the special tax provision so that taxes would not depend on the shape of the lot?

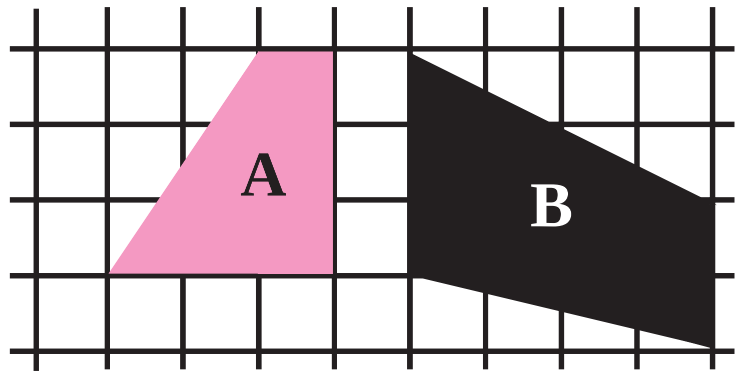
b) What does the tax appraiser mean by "mean parallel side"? by "measurement wide"?

c) Describe in your own words how the tax appraiser calculates the area of a trapezoid.

d) Write as a formula the tax appraiser's rule for calculating the area of a trapezoid that has parallel sides of length a and b if the distance between these sides is d units. Do you think this formula works for a trapezoid that has no right angles? Give a reason for your answer.

❷ a) Is the tax appraiser's rule for calculating the area of a trapezoid the same as the formula you discovered in Activity 2? Explain your answer.

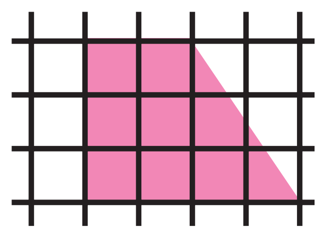
b) Use the tax appraiser's rule to calculate the areas of the trapezoids drawn on this centimetre grid.



❸ a) Draw a line segment to divide trapezoid A in Exercise ❷ into a right triangle and a rectangle. Calculate the areas of the rectangle and triangle to find the area of trapezoid A. Compare with your answer in ❷ b.

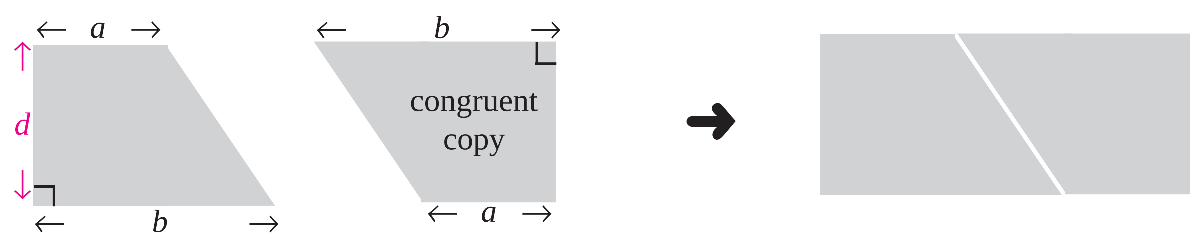
b) Divide trapezoid B in Exercise ❷ into two triangles. Then use the formula for the area of a triangle to calculate the area of trapezoid B. Compare with your answer in Exercise ❷ b.

❹ a) Draw a trapezoid like the one on the right on centimetre paper and count squares to determine its area. Draw another trapezoid congruent to it. Cut out both trapezoids and fit them together to form a rectangle.



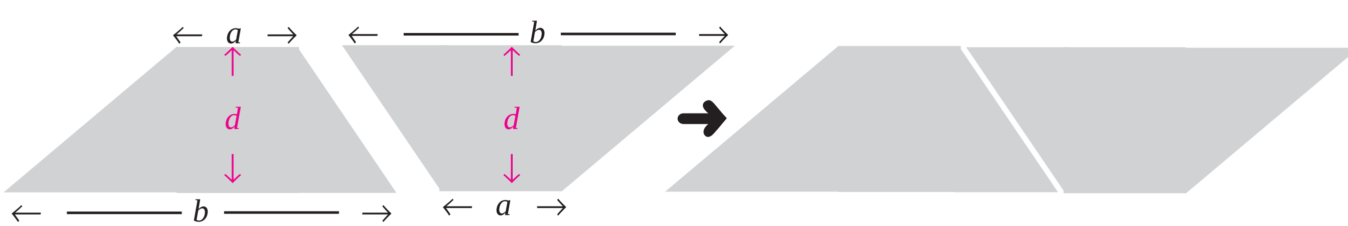
b) Record the area of the rectangle and the area of each trapezoid in ❹ a.

c) A congruent copy of the trapezoid below is made and they are fitted together to form a rectangle as shown.



Write an expression for the area of the rectangle and for the area of each trapezoid in terms of *a*, *b*, and *d*.

d) A congruent copy of the trapezoid below is made and they are fitted together to form a parallelogram as shown.



CHALLENGE

Write an expression for the area of the parallelogram and for the area of the trapezoid in terms of *a*, *b*, and *d*. Show your work.

Grade 7
Answer Key for Activity 3

❶ a) The king based the tax assessment on the area of the lot instead of its shape, so only the size of the lot would matter.

b) The *mean parallel side* is the mean of the lengths of the two parallel sides of a trapezoid. *Measurement wide* means the distance between parallel sides.

c) The tax appraiser adds the lengths of the two parallel sides of a trapezoid and divides by 2 to obtain the mean length of the parallel sides. Then he multiplies by the distance between the parallel sides.

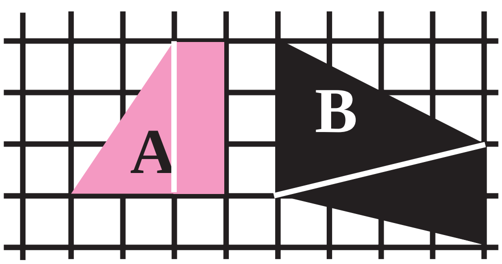
d) $Area = \left(\frac{a+b}{2}\right) \times d$

❷ a) Yes, the sum of the lengths of the parallel sides divided by 2 is the mean parallel side.

b) Area of A = $2 \times 3 = 6$ cm^2.

Area of B = $5/2 \times 4 = 10$ cm^2.

❸ a)



Area of A = Area of rectangle + area of triangle
= 3 cm^2 + 3 cm^2
= 6 cm^2

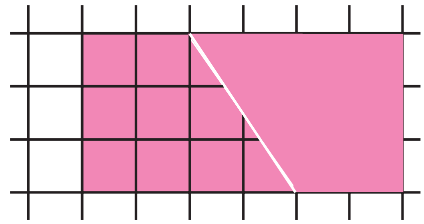
This is the same as the area calculated in ❷ b.

b) We draw a diagonal of the trapezoid B to divide it into two triangles.

Area of B = Area of triangle 1 + Area of triangle 2
= $1/2(3 \times 4)$ cm^2 + $1/2(2 \times 4)$ cm^2
= 6 cm^2 + 4 cm^2
= 10 cm^2.

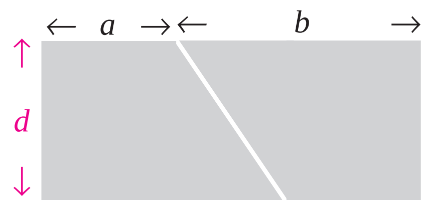
Alternatively, we could have drawn the other diagonal and achieved the same result. This area matches that found in ❷ b.

❹ a) The area of the given trapezoid is 9 cm^2.



b) The area of the rectangle in ❹a is 6 cm $\times$ 3 cm or 18 cm^2. Therefore the area of each half is 9 cm^2 and this is the same as our answer in ❹a.

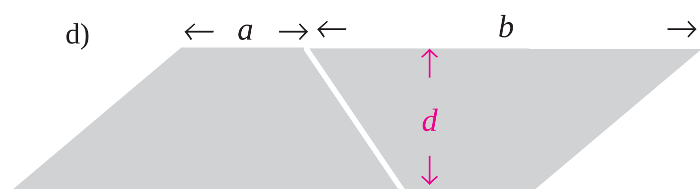
c)



The length of the rectangle is $a+b$.
The width of the rectangle is d.
The area of the rectangle is $(a+b) \times d$.
The trapezoids are congruent and therefore equal in area. The area of each trapezoid is half the area of the rectangle, i.e.,

$$Area = \frac{(a+b) \times d}{2} \quad \text{or} \quad Area = \left(\frac{a+b}{2}\right) \times d$$

d)



The base of the parallelogram is $a+b$.
The height of the parallelogram is d.
The area of the parallelogram is $(a+b) \times d$.
The trapezoids are congruent and therefore equal in area. The area of each trapezoid is half the area of the parallelogram, i.e.,

$$Area = \frac{(a+b) \times d}{2} \quad \text{or} \quad Area = \left(\frac{a+b}{2}\right) \times d$$

The scoring guide presented below has been developed using student responses on a field test conducted in 1998. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 3

	Level 1	Level 2	Level 3	Level 4
APPLICATION **Write the Formula for the Area of a Trapezoid and Apply it (exercise ❹)** **M 7-7, M 7-8**	• Formula is incorrect and improperly applied.	• Formula is correct but improperly applied.	• Formula is correct but minor errors are evident in its application.	• Formula is correct and correctly applied.
COMMUNICATION **Articulation of a Procedure for Calculating the Area of a Trapezoid** (exercises ❶ and ❷ a) **M 7-7, M 7-8**	• Responses to the questions posed are seldom complete and clear. • Responses are inconsistent in their use of appropriate mathematical terminology.	• Responses to the questions posed are often complete and clear. • Responses are inconsistent in their use of appropriate mathematical terminology.	• Responses to the questions posed are complete and clear. • Responses include appropriate mathematical terminology.	• In addition to Level 3: the responses contain clear articulation of the ideas involved and are supported by diagrams, formulas, or examples.

WHAT YOU MIGHT SEE

Communication : Articulation of a Procedure for Calculating the Area of a Trapezoid

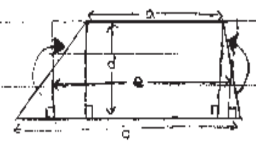
Level 4

Activity 3

1. a) The king revised the tax so that tax would depend on area, not on angles.

← a clear and concise response

b) I think it means the average parallel side.



$(a+b) \div 2 = e$

Since e is kind of a average of the 2 parallel sides, I think mean parallel side is e.

← correct response supported with a diagram, a formula, and the introduction of the symbol *e* for clarification.

↑ The student has reversed the procedure of Activity 2 and transformed a trapezoid into a rectangle.

c) ① He makes a perpendicular line going up from line b to the top left corner in line a.

② He finds half of the distance from the bottom left corner to the perpendicular line.

③ He draws a line going up in 90°.

④ He turnes the little triangle he made by drawing ③ 180°

⑤ He places it on top of the spape made from ③ and ①. This makes a square at the left side.

⑥ He repeats the steps ①~⑤ only this time it's top and bottom right corner and right side. This makes a rectangle.

⑦ Now that he has a rectangle, he needs to know the length of it which is — $(a+b) \div 2 =$ length of the new rectangle.

⑧ Now he take that length and multiply it by the highth of the rectangle to get the area.

← a detailed step-by-step explanation of the tax appraiser's method for calculating the area of a trapezoid.

d) $(a+b) \div 2 \times d$ ← This is a correct formulation of the tax appraiser's rule.

2. a) No it was different. The tax appraiser's rule might be more easier for people who doesn't know my rule. My rule, you can do it faster. $(a+b) \times d \div 2$ ← This is the formula that this student developed in Activity 2.

All the responses are complete and clear. In several cases they are supported by diagrams and formulas. The student has not only described how the tax appraiser calculates the area of a trapezoid, but she has also shown why the formula works by reversing the process in Activity 2 that transforms a rectangle into a trapezoid. The arrows she drew in her diagram indicate congruent triangles and show that the area gained in the transformation is equal to the area lost.

The student has also observed that the tax appraiser's method applies the operations in slightly different order than that discovered in Activity 2, but she acknowledges that both approaches yield the same result.

WHAT YOU MIGHT SEE

Application: Write the Formula for the Area of a Trapezoid and Apply It

Level 2

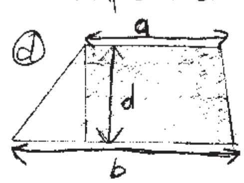
4 b)

Shape	Area
Rectangle	$30\ cm^2$
trapezoid #1	$15\ cm^2$
trapezoid #2	$15\ cm^2$

c) Rectangle $A = (a+b) \times d$

trapezoid $A = (a+b) \times d \div \frac{1}{2}$

d)



$\text{Area} = (a+b) \times d \div \frac{1}{2}$

This student has been able to write correctly the formulas for the area of a rectangle and the general trapezoid. The application of the formula yielded the incorrect area of 30 cm^2 for the rectangle, hence the student obtained the incorrect areas for trapezoids #1 and #2. Since we have only one exercise to judge, it is not clear whether the student made a minor slip or whether there is a more serious problem.

However, it is clear that the student did not estimate the area to check whether the answer was reasonable. A brief chat with the student would enable the teacher to assess the student's difficulty and provide an opportunity to encourage the student to estimate.

Level 4

4 a) Done.

b)

	Area
Tapezoid A	$9 cm^2$
Tapezoid B	$9 cm^2$
Rectangle	$18 cm^2$

c) Rectangle — $(a+b) \times d$ Trapezoid — $(a+b) \times d \div 2$

Trapezoid's congruent copy — $(a+b) \times d \div 2$

d) $(a+b) \times d \div 2$

The student has written the correct formulas for the areas of the rectangle and the trapezoids. These have also been applied correctly. In all the student's responses to Activity 3, it is clear that there is an understanding of why the formula is true and of how to apply it. Of particular note is the student's ability to move from specific numerical examples to the formulation of an algebraic expression in the general case.

Activity 4 – Teacher Edition

Is It Mathematics or Magic?

Expectations Addressed

M 7-2 describe measurement concepts using appropriate measurement vocabulary.

M 7-3 research and report on uses of measurement instruments in projects at home, in the workplace, and in the community.

M 7-6 estimate and calculate the perimeter and area of an irregular two-dimensional shape (e.g., trapezoid, hexagon).

M 7-8 estimate and calculate the area of a trapezoid, using a formula.

Context

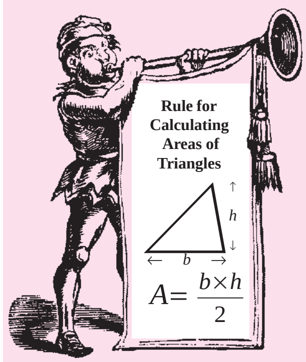
The ongoing battle of wits between the tax appraiser and the mathematicians reaches a climax in this activity. Reacting to the tax appraiser's announcement that tax will be levied on lots according to their areas, the mathematicians decide to exploit the tax appraiser's pride in his area formulas. Knowing that he will assess tax by computing the area of each lot, the mathematicians cleverly rearrange the lots in a rectangle that has an area of 65 square units. The sum of the areas of the lots is 64 square units, so the mathematicians are actually sharing an extra unit of area on which they are not taxed. The tax appraiser knows he has been outmanoevered, but he is unable to resolve the paradox. Students are invited to cut the lots out of centimetre paper and reassemble them into a rectangle to discover where the extra unit of area came from.

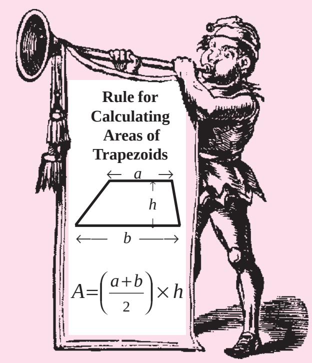
Students are then asked to write a report to the king indicating whether the mathematicians should be taxed on 64 square units in total or 65 square units. They must also include in that report their suggestions about the portion of tax that should be assessed on each lot.

Activity 4 – Student Page

Is It Mathematics or Magic?

We learned in Activity 3 that the tax appraiser in Laputa was very good at calculating areas. He was particularly proud of his rules for calculating the areas of triangles and trapezoids.



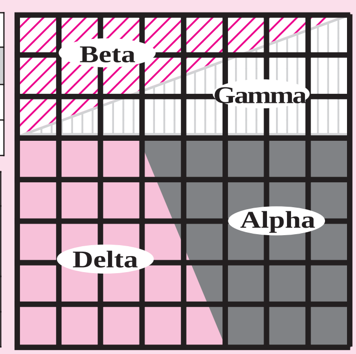


Knowing the tax appraiser's eagerness to apply these rules, the mathematicians Alpha, Beta, Gamma, and Delta constructed their lots as shown here. Each centimetre on the grid stands for a Laputian distance unit.

The tax appraiser recorded the dimensions of each lot in tables like these.

Triangles			
	Base	Height	Area
Beta			
Gamma			

Trapezoids			
	Lengths of Parallel Sides	Distance between Parallel Sides	Area in Square Units
Alpha			
Delta			



Exercise ❶ of this activity has students calculate the areas of lots Alpha, Beta, Gamma, and Delta using the area formulas for triangles and trapezoids. Then in Exercise ❷, the students add the areas obtained in ❶ and discover that the total area is 64 cm^2. The students compare this with their total area when arranged in a rectangle of area 65 cm^2 and are asked to explain why the tax appraiser is confused. The students are confronted with the paradox and asked to verbalize it. Most students are able to recognize that there is a paradox, but unable to explain why it occurs. Exercise ❸ provides students with an opportunity to cut the lots out of centimetre paper and discover for themselves that the pieces do fit together to form a square but they do not quite fit together to form a rectangle. A small sliver of space with an area of (you guessed it!) one square centimetre appears along the main diagonal and once again area is conserved!

The Lesson Launch

3 minutes

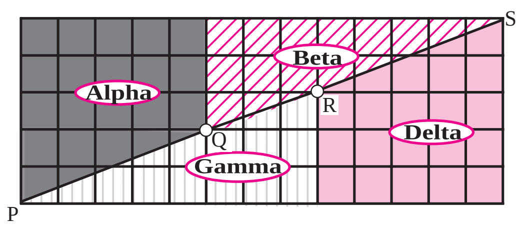
Distribute copies of pages 48 and 49, two sheets of centimetre paper (see p. 56), and scissors to all students. Remind them that the tax appraiser will now tax all lots in proportion to their areas. Furthermore, as we discovered in Activity 3, the tax appraiser's rules for calculating areas of trapezoids and triangles enable him to compute the tax owing with high precision. What will the mathematicians do next?

Initiating Activity

30 minutes

Group the students in pairs and assign Exercises ❶ – ❸. Observe students as they enter the areas of the lots into the tables to ensure that they are applying the formulas correctly. It is also appropriate for students to determine the areas by decomposing trapezoids into rectangles and right triangles.

As the students progress through the exercises, you will see them cutting out the pieces representing the four lots. The most efficient way to do this is to cut an 8×8 square out of centimetre paper and then cut that square into the four lots. As the students arrange the lots into a rectangle, some will observe that there is a small sliver of space along the diagonal. This is a consequence of the fact that lots Beta and Gamma in the rectangular diagram are *not* triangles!



Observe that the slope of PQ is 2/5 or 0.4. Similarly the slope of line segment RS is 2/5 or 0.4. However, the slope of QR is 1/3 or 0.33. That is, points P, Q, and R are not on the same line segment, so lot Gamma is not a triangle. Similarly, Beta is not a triangle, so we cannot use the area formula for triangles in computing the areas of lots Beta and Gamma.

Cooperative Learning Activity

25 minutes

Partition the class into groups of four or five. Appoint a chair, a recorder, and a reporter for each group. Each group is to prepare a report to the king as outlined on page 49.

Activity 4 – Student Page

❶ a) Complete the tables page 48, with each centimetre a Laputian unit.

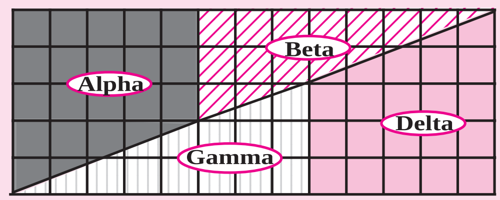
b) What do you notice about the areas of triangles Beta and Gamma?

c) Are Beta and Gamma congruent triangles? Explain why or why not.

d) Are trapezoids Alpha and Delta congruent? Explain why or why not.

e) Add the areas in your table to find the total area of all four lots.

After the tax appraiser computed the areas of the four lots, the mathematicians rearranged their lots on the building plans as shown below.



❷ a) What is the total area of this rectangle?

b) Compare your answers in ❶e and ❷a and explain why the tax appraiser became confused.

❸ a) Using centimetre paper, cut out lots Alpha, Beta, Gamma, and Delta with the dimensions given in the table on page 48. Show that they have a total area of 64 cm^2 by arranging them in an 8 cm $\times$ 8 cm square.

b) Arrange these lots in a rectangle of length 13 cm and width 5 cm. What is the total area of the lots? Where did the extra area come from?

Report Write a letter on a scroll to the king indicating whether the total area taxed should be 64 square units or 65 square units. Give reasons to support your argument.

Indicate what percent of the total tax should be assigned to each of the four lots.

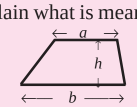
Research:
How is land taxed in your municipality?
By area? By frontage?
By market value?

Tax Appraiser's Limerick

Challenge

The trapezoid rule 'tis true,
Applies to other shapes too.
Triangle's trapezoid
With one side that's void,
And parallelogram follows the rule.

The tax appraiser's limerick suggests that the formula for the area of a trapezoid applies to triangles and parallelograms. Explain what is meant by "one side that's void." Show how the formula for the area of a trapezoid becomes



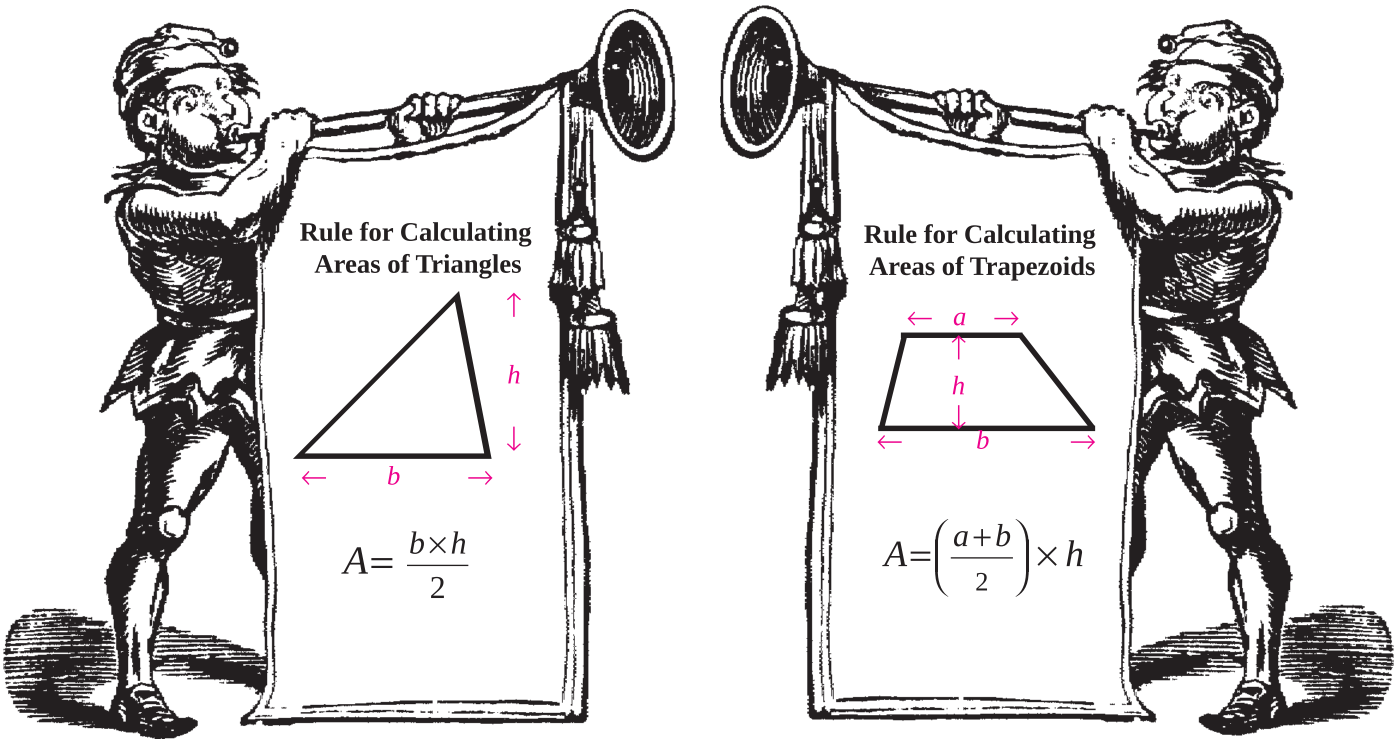
a) $b \times h/2$ as side length a gets close to 0.
b) $b \times h$ when $a = b$.

Closure

When the students are finished, invite each group to present its report. Challenge reporters presenting opposing opinions to defend their positions. Promote discussion and pose questions that lead to the realization that lots Beta and Gamma in the rectangle are not triangles, but quadrilaterals. Have the students calculate the areas of lots Beta and Gamma to verify that the total area is 65 cm^2. Finish the lesson by discussing the meaning of the Tax Appraiser's limerick. Show students that as the side of length a shrinks to 0 ("becomes void"), the area formula for a trapezoid becomes the area formula for a triangle, and as a increases to b, it becomes the area formula for a parallelogram.

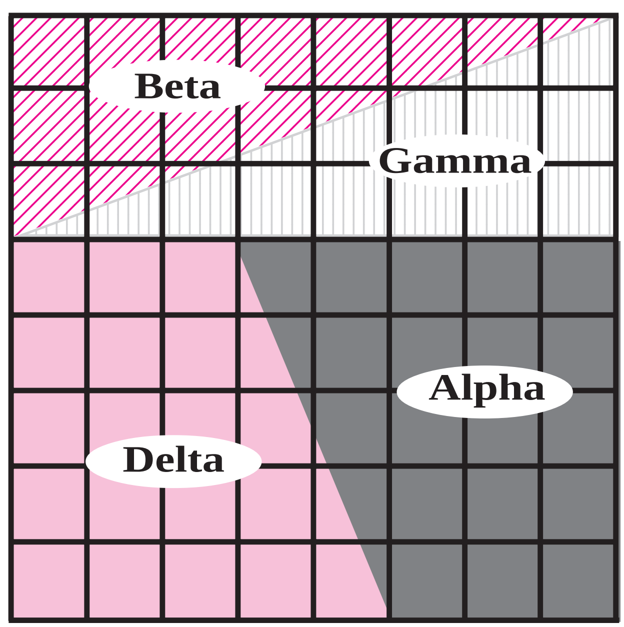
Is It Mathematics or Magic?

We learned in Activity 3 that the tax appraiser in Laputa was very good at calculating areas. He was particularly proud of his rules for calculating the areas of triangles and trapezoids.



Knowing the tax appraiser's eagerness to apply these rules, the mathematicians Alpha, Beta, Gamma, and Delta constructed their lots as shown here. Each centimetre on the grid stands for a Laputian distance unit.

The tax appraiser recorded the dimensions of each lot in tables like these.



Triangles			
	Base	Height	Area
Beta			
Gamma			

Trapezoids			
	Lengths of Parallel Sides	Distance between Parallel Sides	Area in Square Units
Alpha			
Delta			

❶ a) Complete the tables on the other page, recording each centimetre as a Laputian unit.

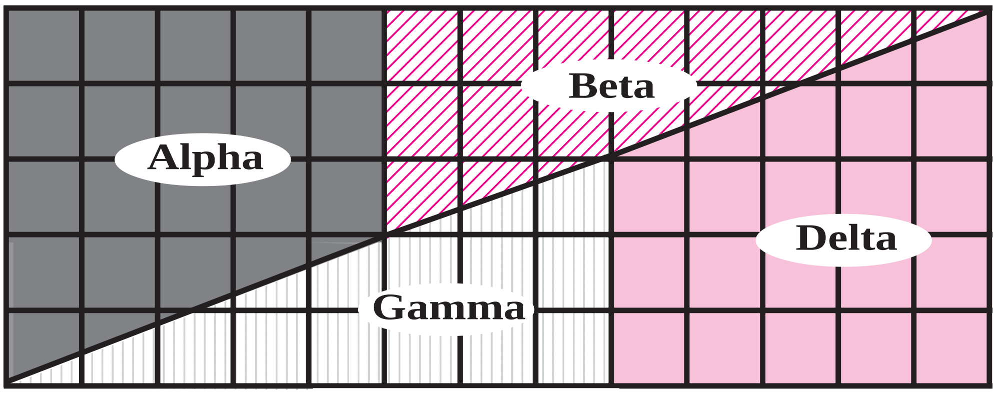
b) What do you notice about the areas of triangles Beta and Gamma?

c) Are Beta and Gamma congruent triangles? Explain why or why not.

d) Are trapezoids Alpha and Delta congruent? Explain why or why not.

e) Add the areas in your table to find the total area of all four lots.

After the tax appraiser computed the areas of the four lots, the mathematicians rearranged their lots on the building plans as shown below.



❷ a) What is the total area of this rectangle?

b) Compare your answers in ❶ e and ❷ a and explain why the tax appraiser became confused.

❸ a) Using centimetre paper, cut out lots Alpha, Beta, Gamma, and Delta with the dimensions given in your tables. Show they have a total area of 64 cm^2 by arranging them in an 8 cm × 8 cm square.

b) Arrange these lots in a rectangle of length 13 cm and width 5 cm. What is the total area of the lots? Where did the extra unit of area come from?

REPORT

Write a letter on a scroll to the king indicating whether the total area taxed should be 64 square units or 65 square units. Give reasons to support your argument.

Indicate what percent of the total tax should be assigned to each of the four lots.

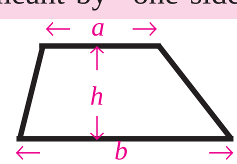
Research:
How is land taxed in your municipality?
By area? By frontage?
By market value?

CHALLENGE

TAX APPRAISER'S LIMERICK

The trapezoid rule 'tis true,
Applies to other shapes too.
Triangle's trapezoid
With one side that's void,
And parallelogram follows the rule.

The tax appraiser's limerick suggests that the formula for the area of a trapezoid applies to triangles and parallelograms. Explain what is meant by "one side that's void." Show how the formula for the area of a trapezoid becomes a) $b \times h/2$ as side length a gets close to 0.
b) $b \times h$ when $a = b$.



GRADE 7
ANSWER KEY FOR ACTIVITY 4

❶ a) The completed tables are shown below.

Triangles			
	Base	Height	Area
Beta	8	3	12
Gamma	8	3	12

Trapezoids			
	Lengths of Parallel Sides	Distance between Parallel Sides	Area in Square Units
Alpha	3, 5	5	20
Delta	3, 5	5	20

b) The areas of Beta and Gamma are equal.

c) Beta and Gamma are right triangles with legs of length 8 and 3. Two such right triangles are congruent because they have two sides and a contained angle respectively equal. Alternatively, students may observe that they are congruent because they can be placed in coincidence.

d) Trapezoids Alpha and Delta are congruent because either can be cut out and placed in superposition with the other.

e) The sum of the areas of the four lots as shown in the table is 12 + 12 + 20 + 20 or 64 square units.

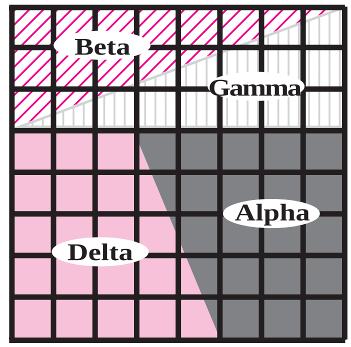
❷ a) The rectangle containing the four lots is 13 units long and 5 units wide, and so has an area of 5×13 or 65 square units.

b) The areas of the lots given in the table have a total of 64 square units, but the rectangle formed by the lots has an area of 65 square units. Therefore the tax collector does not know whether to assess an area of 64 square units or 65 square units. That is why he was confused.

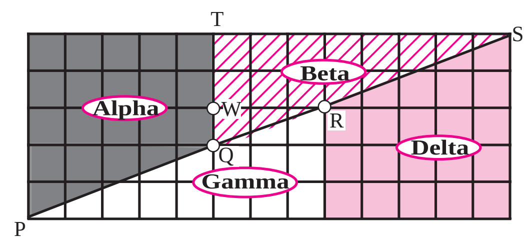
TEACHER NOTE

The apparent paradox presented above is a classic visual trick. When the lots are arranged in an 8×8 square, they fit together without spaces or overlap. However, when rearranged to form a 13×5 rectangle, the lots *appear* to fit together but, as students may discover in Exercise ❸, there is a small gap along the main diagonal that has a total area of one square unit.

❸ a) The students will display this:



b) When the students put the pieces together to form a rectangle, there will be a sliver of space of area 1 cm^2 along the main diagonal. The reason for this is that when the rectangle is partitioned into lots Alpha, Beta, Gamma, and Delta, these lots are all quadrilaterals. Beta and Gamma are not triangles because line segments PQ, QR, and RS have different slopes. To find the areas of lots Beta and Gamma, we must break each into a trapezoid plus a triangle.



Area of lot Beta

= Area of triangle QWR + Area of trapezoid TWRS

= 3/2 cm^2 + 11/2 × 2 cm^2.

= 12.5 cm^2

Similarly, the area of lot Gamma = 12.5 cm^2. Together, lots Beta and Gamma have a combined area of 25 cm^2, which is 1 cm more than the triangular lots Beta and Gamma. The extra 1 cm^2 came from reshaping the triangular lots Beta and Gamma into quadrilaterals of slightly larger area.

TAX APPRAISER'S LIMERICK

The area formulas for the triangle and parallelogram are special cases of the area formula for trapezoids. The idea that formulas remain unchanged as one shape changes into another is a powerful concept that pervades mathematics and motivates the continual quest for general forms. At this level it's just an interesting curiosity for kids.

The scoring guide presented below has been developed using student responses on a field test conducted in 1998. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 4

	Level 1	Level 2	Level 3	Level 4
PROBLEM SOLVING **Able to identify & Solve an area Paradox** (exercises ❷, ❸, and Report) **M 7-6, M 7-8**	• shows little awareness that there is a paradox. • little or no awareness that at least one of the lots has increased in size.	• is unaware that at least one of the lots has increased in area. • does not identify 65 square units as the area on which the total tax should be calculated.	• is aware that at least one of the lots has increased in area. • identifies 65 square units as the area on which the total tax should be calculated.	• is aware that at least one of the lots has increased in area. • is able to calculate the correct proportions, 30.77%, 19.23%, 19.23%, and 30.77% of the 65 square units on which tax is to be paid.
COMMUNICATION **Reporting of Recommendations Regarding an Area paradox** (the Report)	• responses to the questions posed are missing and/or unreasonable. • reasons supporting some of the recommendations are missing and/or incoherent.	• responses to the questions posed are complete and clear. • reasons supporting some of the recommendations are missing and/or incoherent.	• responses to the questions posed are complete and clear. • reasons supporting the recommendations are clear and coherent.	• in addition to Level 3: the report contains clear articulation of the ideas involved and offers persuasive support for the position taken.

WHAT YOU MIGHT SEE

Problem Solving: Ability to Identify & Resolve an Area Paradox

Level 1

REPORT

Dear King,

You should tax theirs lot's 64 unit² because if you add the sections seperatly you will find that it will be 64 unit². Also if you arrange it into a square it will be 64 unit².

Beta 20%
Gamma 20%
Delta 30%
Alpha 30%

This report contains nothing suggesting that the student is aware of an area paradox. In his answer to Exercise ❷ b (not shown here), asking why the tax appraiser became confused, the student answered, "He became confused because 2 shapes had areas of 12 and the other 2 had an area of 20."

There is the observation that the total area of the lots is 64 square units and the recommendation that the tax be based on this area and levied in the proportions 20%, 20%, 30%, 30%. There is no explanation or displayed computation showing how these percents were obtained.

Level 2

2 a) It is 65 unit²

b) The tax appraiser got confused because one is 65 unit² and one is 64 unit² but the land area is the same.

In page 49 2 different shapes (trapezoid & triangle) are put together to make a big triangle. The thing about triangles is that if you have one triangle you need another as big to make a rectangle or a square. In page 48 they fit 2 trapezoids and triangles into rectangles. And if you put these 2 rectangles together you get a square. The other one takes up more space by having a long triangle while this one is all nicely packed up.

Report

Your Majesty .....

The total tax for Beta, Gamma, Delta, and Alpha has 2 different areas. One is 65unit² and the other is 64unit². I think the area should be taxed 64 unit² because they had cleverly tricked me. They had found a way to get less tax but it doesn't break any of your rules. And this time you can't do anything about it.

P.S. Beta - 18.75%
Gamma - 18.75%
Delta - 31.25%
Alpha - 31.25%

This response to Exercise ❷ b shows that the student is aware that lots whose total area is 64 square units seem to have been rearranged into a rectangular shape with a total area of 65 square units. The student's detailed explanation of the different configurations on pages 48 and 49 reveals that the student is grappling with the problem, but has not resolved to her own satisfaction why this paradox occurs. She has not realized that lots Beta and Gamma have changed from triangles to quadrilaterals.

In her report, she graciously accepts full responsibility for having been cleverly outmanoevered by the mathematicians, and suggests to the king that he ignore the rearrangement and levy tax on 64 square units. Her recommended apportionment of taxes among the four lot owners is correct if the tax were to be levied on 64 square units.

WHAT YOU MIGHT SEE

PROBLEM SOLVING: ABILITY TO IDENTIFY & RESOLVE AN AREA PARADOX

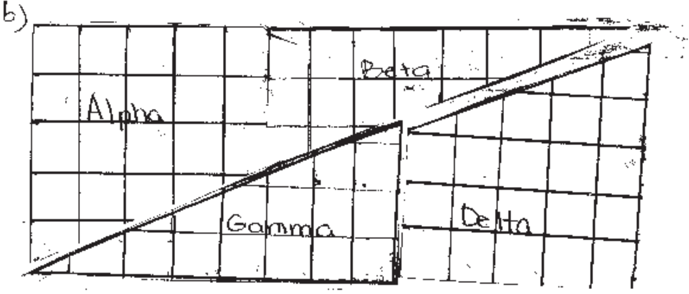
Level 3

cm = Lupition distance unit.

b) a) says the total area of the 4 lots is 64 cm^2, but part b) says the total area is 65 cm^2. There is a difference of 1 cm^2. The tax appraiser must be wondering where the 1 m^2 is missing.

← The student recognizes that there is an area paradox.

③ b)



The extra came from the upper left corner of Delta. In the example, Delta has an extra 1 cm^2 to it. The diagonal line up the rectangle up to the right goes a bit higher on delta, not making a huge difference, but made delta a bit bigger.

← The student realizes that at least one of the lots must have increased in area. Upon assembling the lots in a rectangle, the student observes a space along the diagonal and assumes (incorrectly) that the space is filled by increasing the size of Delta.

Report

Dear tax appraiser,

To the four lots of the mathematicians, Alpha, Beta, Delta and Gamma, a reasonable solution for your confusion is to make them pay tax for 65 cm^2. Their area is 65 cm^2, for delta has an area of 21 cm^2 not 20 cm^2. The following are the precentage of the amount each lot should pay for:

Alpha : $\frac{20}{65}$ = 31 %

Beta : $\frac{12}{65}$ = 18 %

Gamma : $\frac{12}{65}$ = 18 %

Delta : $\frac{21}{65}$ = 32 %

Delta should pay the most tax at 32 %, Alpha should pay the second most tax at 31 %, Beta and Gamma should pay equal amount at 18 % each of the total tax.

This report correctly suggests that the total area to be taxed is 65 square units. However, the student incorrectly assumes that it is lot Delta that has increased in size. The displayed computations show that the student understands the correct way to determine the area of each lot as a fraction of the total area.

WHAT YOU MIGHT SEE

Problem Solving: Ability to Identify & Resolve an Area Paradox

Level 4

② a) 65 cm^2

b) The line that cut the rectange in half is not striaght.

③ The line which have all sharps attached had curled at two points. Thus caused The two triangles larger than they are measured.

You Highness:
The four clever Mathematics had changed their 64 cm^2 lots into 65 cm^2 lots by change Beta's triangle and Gramma's triangle into four-sided shapes, which made them larger than they are measured.
Their ture lots areas are:
Alpha: 20 cm^2
Beta: 12.5 cm^2
Gramma: 12.5 cm^2
Delta: 20 cm^2
Which is total 65 cm^2.

This report reveals a high level of problem solving ability. The student has realized that the diagonal of the rectangle is not a straight line. This is a significant achievement for a student who has not yet learned about slopes of line segments.

This student has also recognized that the two triangles have been transformed into quadrilaterals. She then calculates correctly the areas of these quadrilaterals, presumably by decomposition into a rectangle and triangle.

She stops short of an explicit allocation of tax responsibility to each lot owner. Perhaps she felt that the hard work was done and the rest is trivial detail. She is right.

Communication: Reporting of Recommendations Regarding an Area Paradox

Level 1

Report Sept, 30, 98

Using the 65 square units to calulate the total tax on the four lot is a good idea. First of all you will get more money and they can't fool you again. Second, the people will have a bit more of a lot. Beta and Gamma should be assigned 30% of the taxes and Alpha and Delta should be assigned 70% of the taxs because they have bigger lots.

This report addresses the questions posed but the second reason given for recommending assessment on 65 square units is incoherent. The statement “the people will have a bit more of a lot,” needs clarification.

The report recommends a combined 30% assessment for those with the smaller lots and a combined 70% assessment for those with the larger lots. These percentages are inappropriate for either the 64 or 65 square unit assessments. Furthermore, there is no displayed calculation or stated procedure explaining how these percentages were obtained.

WHAT YOU MIGHT SEE

Communication: Reporting of Recommendations Regarding an Area Paradox

Level 2

Report

We suggest that the tax appraiser uses $64 cm^2$ to tax Alpha, Betta, Delta and Gamma, because the areas of each lot added together is $64 cm^2$. Whatever the tax is, Alpha and Delta should be charged: 31.25% each, and Betta and Gamma should be charged: 18.75% each.

This report responds completely and clearly to the questions posed, and the recommendations are reasonable. However, the reasons supporting the tax allotments are not given explicitly.

Level 2

You Highness:
The four clever Mathematics had changed their $64 cm^2$ lots into $65 cm^2$ lots by change Beta's triangle and Gramma's triangle into four-sided shapes, which made them larger than they are measured.
Their ture lots areas are:
Alpha: $20 cm^2$
Beta: $12.5 cm^2$
Gramma: $12.5 cm^2$
Delta: $20 cm^2$
Which is total $65 cm^2$.

A strict application of the rubric would assign this report a Level 1, since it does not explicitly state a recommendation for the allocation of tax to lots. However, it is clear that the student intends allocation by area. No displayed computational support for the areas quoted has been provided.

Level 3

Report

Dear tax appraiser,

To the four lots of the mathematicians, alpha, Beta, Delta and Gamma, a reasonable solution for your confusion is to make them pay tax for $65 cm^2$. Their area is $65 cm^2$, for delta has an area of $21 cm^2$ not $20 cm^2$. The following are the precentage of the amount each lot should pay for:

Alpha : $\frac{20}{65}$ = 31 %

Beta : $\frac{12}{65}$ = 18 %

Gamma : $\frac{12}{65}$ = 18 %

Delta : $\frac{21}{65}$ = 32 %

Delta should pay the most tax at 32 %, Alpha should pay the second most tax at 31 %, Beta and Gamma should pay equal amount at 18 % each of the total Tax.

Teacher note:
The last two samples on this page, also displayed on pages 53 and 54, show how a sample of student work can represent different achievement levels relative to different skill categories.

This report addresses both questions clearly and completely. The computations are displayed to show how the taxes were assigned to each lot. Although there is an error in the calculation of the area of lot Delta, the student has followed through with consistency.

Template – Centimetre Paper

PART IV

Measurement in Grade 8

Measurement: Grade 8

Overall Expectations

By the end of Grade 8, students will:

- demonstrate a verbal and written understanding of and ability to apply accurate measurement and estimation strategies that relate to their environment;
- identify relationships between and among measurement concepts (linear, square, cubic, temporal, monetary);
- solve problems related to the calculation of the radius, diameter, and circumference of a circle;
- apply volume and area formulas to problem-solving situations involving triangular prisms.

Specific Expectations

For convenient reference, the specific expectations are coded. M 8-1 refers to the first Measurement expectation in Grade 8.

Students will:

Units of Measure

M 8-1 - use listening, reading, and viewing skills to interpret and evaluate the use of measurement formulas;

M 8-2 - explain the relationships between various units of measurement;

M 8-3 - research, describe, and report on uses of measurement in projects at home, in the workplace, and in the community that require precise measurements;

M 8-4 - make increasingly more informed and accurate measurement estimations based on an understanding of formulas and the results of investigations;

M 8-5 - ask questions to clarify and extend their knowledge of linear measurement, area, volume, capacity, and mass, using appropriate measurement vocabulary;

Specific Expectations (cont'd)

Students will:

Perimeter, Circumference, and Area

M 8-6 - measure the radius, diameter, and circumference of a circle using concrete materials;

M 8-7 - recognize that there is a constant relationship between the radius, diameter, and circumference of a circle, and approximate its value through investigation;

M 8-8 - develop the formula for finding the circumference and the formula for finding the area of a circle;

M 8-9 - estimate and calculate the radius, diameter, circumference, and area of a circle, using a formula in a problem-solving context;

M 8-10 - draw a circle given its area and/or circumference;

M 8-11 - define radius, diameter, and circumference and explain the relationships between them;

M 8-12 - develop the formula for finding the surface area of a triangular prism using nets;

Capacity, Volume, and Mass

M 8-13 - develop the formula for finding the volume of a triangular prism (area of base × height);

M 8-14 - understand the relationship between the dimensions and the volume of a triangular prism;

M 8-15 - calculate the surface area and the volume of a triangular prism, using a formula in a problem-solving context;

M 8-16 - sketch a triangular prism given its volume.

Activity 1 – Teacher Edition

The Ingenious Beverage Can – Will We Ever Run Out?

Expectations Addressed

M 8-4 make increasingly more informed and accurate measurement estimations based on an understanding of formulas and the results of investigations.

M 8-6 measure the radius, diameter, and circumference of a circle using concrete materials.

M 8-7 recognize that there is a constant relationship between the radius, diameter, and circumference of a circle, and approximate its value through investigation.

M 8-8 develop the formula for finding the circumference and the formula for finding the area of a circle.

M 8-9 estimate and calculate the radius, diameter, circumference, and area of a circle, using a formula in a problem-solving context.

M 8-10 draw a circle given its area and/or circumference.

M 8-11 define radius, diameter, and circumference and explain the relationships between them.

DM 8-18 determine trends and patterns by making inferences from graphs.

Context

This unit embeds the investigation of the circumference and area properties of circles in the rich context of commercial packaging– in particular, the evolution of the beverage can. This reinforces the Structural Strength and Stability strand of *The Ontario Curriculum, Grades 1–8: Science and Technology* which includes the following expectation on p. 85:

- *tell the "story" of a product used every day, identifying the need it meets and describing its production, use, and eventual disposal.*

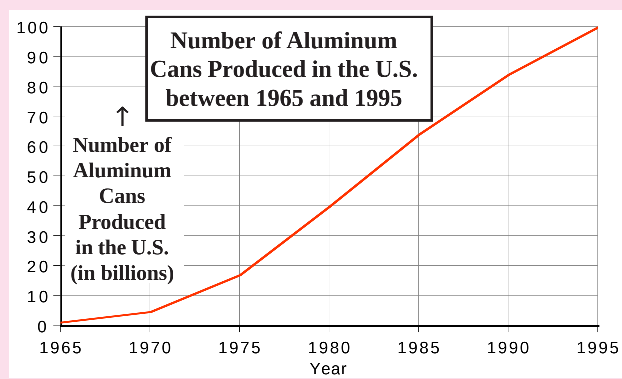
Activity 1 presents students with a brief history of the soft drink industry and the emergence of the aluminum beverage can. A line graph showing the number of aluminum cans produced in the U.S. between 1965 and 1995 reviews graphing concepts. By measuring the diameters and circumferences of various cans, students are led to discover the C = πD relationship for circles. In Activity 2, they investigate the formula for the area of a circle to determine how much wastage results when circular disks are punched out of a sheet of aluminum. Activity 3 uses the C = πD and A = $\pi D^2/4$ relationships to compare the surface areas of cylindrical cans of the same volume but different height and diameter. Students investigate the height:diameter ratio that is optimal in minimizing the aluminum needed for a beverage can of capacity 355mL. Finally, in Activity 4, students investigate the amount of aluminum saved by recycling and are led through calculations and Internet investigations to the discovery that conservation is ultimately the most viable way to save the aluminum can from extinction.

Activity 1 – Student Page

The Ingenious Beverage Can–Will We Ever Run Out?

The invention of Coca Cola in 1886, and the establishment of Pepsi Cola in 1898 mark the beginning of the ever-expanding soft drink industry. Since then, the original recipes for Coke and Pepsi have been modified and a wide variety of soft drinks have emerged. But even more remarkable than the emergence of soft drinks has been the development of the containers that hold them.

Originally, soft drinks were sold in returnable bottles. Then in 1935, steel beverage cans appeared, challenging the returnable bottle as the basic beverage container. By 1967, Coca Cola and PepsiCo were selling soft drinks in aluminum beverage cans and a new era in beverage containers was born. Since then the demand for aluminum beverage cans has exploded, as shown in the graph below.



Graph Literacy

❶ Use your graph to help you answer these questions.

a) About how many aluminum beverage cans were produced in 1980?
b) About what year was the production of cans double that of 1980?
c) How many times as great as in 1976 was the production of cans in 1995?
d) In about what year were 70 000 000 000 cans produced?
e) Did the number of cans produced in the U.S. increase by a constant amount each year?
f) Is it possible to determine from the graph how many years it takes for the production of cans to double? Explain.

The Lesson Launch 5 minutes

Before launching this lesson, read the article by Hosford and Duncan (see reference p. 96). It is a rich source of information on the remarkable engineering involved in the development of the aluminum beverage can.

Hold up a couple of different soft drink cans and pose to the class questions such as the following:

- *What material are these cans made of?*
- *Where do we get this material? Is there an unlimited amount of it?* (Answer: Aluminum does not occur as an element in nature but rather as a compound from which the aluminum must be chemically extracted at considerable cost.)
- *About how many aluminum cans do you think were produced in the U.S. in 1995?*

Use the information gleaned from the Hosford and Duncan article to describe to the students some of the remarkable engineering feats achieved in developing the can.

Initiating Activity 10 minutes

Distribute copies of student page 62. Have students work individually to complete Exercise ❶ in their notebooks. When the students are finished, display on the overhead projector the graph on page 62. Invite students to present their answers to the questions in that exercise.

Paired Activity 30 minutes

Group students in pairs and provide each pair with four cans of different diameters, a tape measure calibrated in millimetres, a copy of page 63, and a sheet of centimetre paper (see template p. 56). Discuss briefly the terms *diameter, radius, circumference* and *surface area*. Have students complete Exercises ❶–❻ on page 63, ensuring both students in each pair make their own table and graph. If they have access to a spreadsheet or graphing calculator, encourage them to use it in Exercise ❺ to construct the table and scatterplot.

As you circulate around the room, check that the numbers in the C/D column of the students' tables are close to π. Check also that the points on the graph lie close to the line of slope π that passes through the origin.

Activity 1 – Student Page

❷ What geometric solid resembles the shape of a can?
Explain in a sentence or draw a diagram to show what is meant by each of these measures of a beverage can.
a) its diameter b) its circumference
c) its surface area d) its volume

❸ Describe how you would measure the diameter, circumference, and height of a can.

❹ a) Without measuring, estimate which is greater, the circumference of a beverage can or its height.

b) Estimate how many times as great as the diameter is the circumference.

❺ a) Measure and record the diameter, circumference, and height of each can. Check your estimates in Exercise ❹. Were you surprised? Explain.

b) Enter in a spreadsheet or in a table like this your measures of the diameter and circumference of each can. Calculate C/D for each can and enter its value into your table.

	Circumference C	Diameter D	C/D
CAN #1			
CAN #2			
CAN #3			
CAN #4			

c) Make a graph of C vs. D on your centimetre paper or using the *graph* or *chart* menu on your spreadsheet. What does your graph suggest about the relationship between circumference and diameter? Explain.

d) Describe any pattern in the numbers in the C/D column of your table. Use this pattern to write a formula for C in terms of D.

e) Use your formula to calculate the circumference of a can of diameter 10 cm.

f) Use your formula to calculate the diameter of a can with circumference 25 cm.

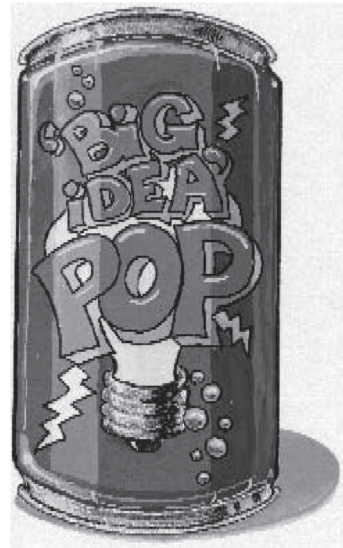
❻ Describe three ways that engineers might design aluminum cans so that they use less aluminum.

Closure

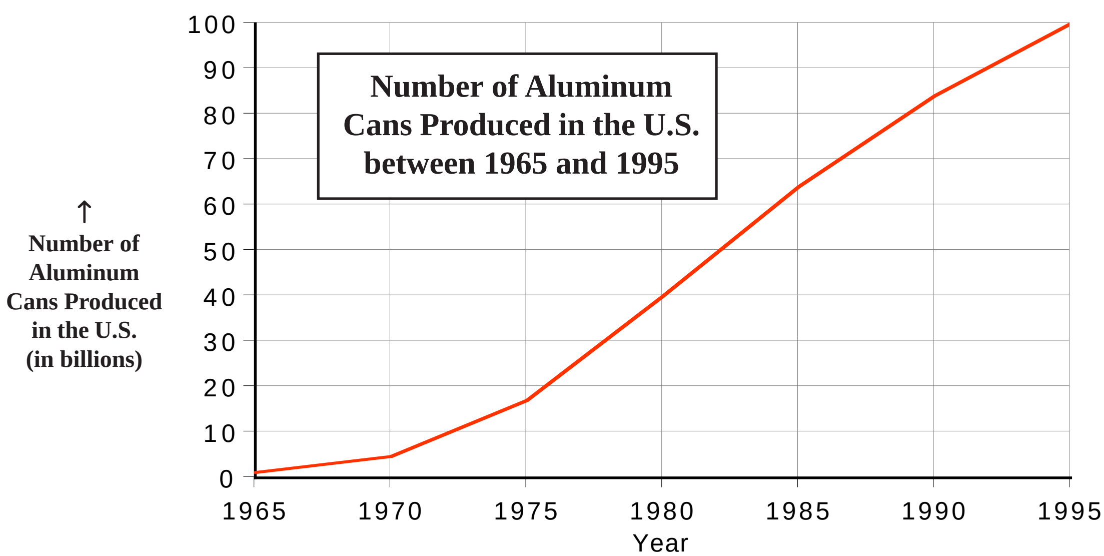
When all the students have finished, ask whether the beverage can is taller or wider around. Many will be surprised to discover that the circumference of a beverage can is much greater than its height. Reinforce this by having students attempt to wrap their thumb and forefinger *around* a beverage can. Then have them span the height of the can with thumb and forefinger. Remind them that the circumference of a circle is about three times its diameter. Then call out different diameters and ask them to estimate the corresponding circumferences. Then call out the circumferences of different circles and ask them to estimate the corresponding diameters. After reinforcing the approximate 3 to 1 relationship between circumference and diameter, formalize it by introducing π and explaining that it's just a symbol that stands for the exact ratio. Wrap up the lesson with a brief discussion of different ways to reduce aluminum use in cans.

The Ingenious Beverage Can – Will We Ever Run Out?

The invention of Coca Cola in 1886, and the establishment of Pepsi Cola in 1898 mark the beginning of the ever-expanding soft drink industry. Since then, the original recipes for Coke and Pepsi have been modified and a wide variety of soft drinks have emerged. But even more remarkable than the emergence of soft drinks has been the development of the containers that hold them.



Originally, soft drinks were sold in returnable bottles. Then in 1935, steel beverage cans appeared, challenging the returnable bottle as the basic beverage container. By 1967, Coca Cola and PepsiCo were selling soft drinks in aluminum beverage cans and a new era in beverage containers was born. Since then the demand for aluminum beverage cans has exploded, as shown in the graph below.



Graph Literacy

❶ Use your graph to help you answer these questions.

a) About how many aluminum beverage cans were produced in 1980?
b) About what year was the production of cans double that of 1980?
c) How many times as great as in 1976 was the production of cans in 1995?
d) In about what year were 70 000 000 000 cans produced?
e) Did the number of cans produced in the U.S. increase by a constant amount each year?
f) Is it possible to determine from the graph how many years it takes for the production of cans to double? Explain.

To do these exercises, you will need four empty cans of different diameters, a measuring tape marked in millimetres, and a sheet of squared paper or access to a spreadsheet.

❷ What geometric solid resembles the shape of a can?
Explain in a sentence or draw a diagram to show what is meant by each of these measures of a beverage can.
a) its diameter b) its circumference c) its surface area d) its volume

❸ Describe how you would measure the diameter, circumference, and height of a can.

❹ a) Without measuring, estimate which is greater, the circumference of a beverage can or its height.
b) Estimate how many times as great as the diameter is the circumference.

❺ a) Measure and record the diameter, circumference, and height of each can. Check your estimates in Exercise ❹. Were you surprised? Explain.
b) Enter in a spreadsheet or in a table like this your measures of the diameter and circumference of each can. Calculate C/D for each can and enter its value into your table.

	Circumference C	**Diameter D**	**C/D**
CAN #1			
CAN #2			
CAN #3			
CAN #4			

c) Make a graph of C vs. D on your centimetre paper or using the *graph* or *chart* menu on your spreadsheet. What does your graph suggest about the relationship between circumference and diameter? Explain.
d) Describe any pattern in the numbers in the C/D column of your table. Use this pattern to write a formula for C in terms of D.
e) Use your formula to calculate the circumference of a can of diameter 10 cm.
f) Use your formula to calculate the diameter of a can with circumference 25 cm.

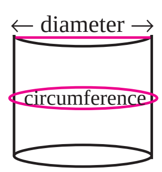
❻ Describe three ways that engineers might design aluminum cans so that they use less aluminum.

Grade 8

Answer Key for Activity 1

❶ a) About 40 billion (40 000 000 000).
b) 1988–89
c) About 5 times or 500%.
d) 1987
e) Between 1970 and 1980 the production of cans increased by about 35 billion and between 1980 and 1990 it increased by about 45 billion. Therefore production did not increase by a constant amount over each decade or over each year. However, the graph between 1975 and 1985 is almost linear indicating that the yearly growth was almost constant during that period.
f) The production of aluminum cans increased from 5 billion in 1970 to 10 billion in 1972. That is, production doubled in 2 years. It took from 1972 to 1976 (four years) for production to double again. Therefore the doubling time depends on the year. Using the graph, we can determine the doubling time for any year up to 1982. After that, the doubling of production is greater than 100 billion cans and this falls beyond the range of data in the graph.

❷ The cylinder is a model for a can.

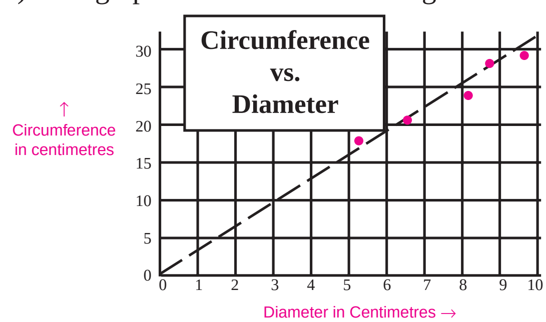


a) The *diameter* of a can is the distance across the top surface of the can measured through the centre of that surface.
b) The *circumference* of a can is the distance around its curved surface.
c) The *surface area of the curved surface* of a can is the area of the rectangle produced by slicing the curved surface vertically and flattening it. The *surface area* of a can is the surface area of the curved surface plus the areas of the top and bottom of the can.
d) The *volume* of a can is the amount of space it occupies.

❸ a) To measure the diameter of a can we measure the distance across its top or bottom surface at the widest point.
b) To measure the circumference, we can use a string and measure the distance around the curved surface of the can.
c) To measure the height of a can, we measure the distance from top to bottom.

❹ a) Students will tend to think that the height is greater or that circumference and height are very close. To demonstrate the relationship, attempt to wrap your thumb and finger around the can and show students that they don't touch. Then show how your outstretched thumb and finger can easily span the height of the can.
b) Few students will recognize that the circumference is more than three times (actually π times) the diameter.

❺ a) Answers will vary.
b) $C/D = \pi$ for all cans. Student measurements should generate values of C/D somewhere between 2.8 and 3.4.
c) The graph should look something like this.



The points on the graph lie close to a straight line. This suggests that circumference is a linear function of diameter.
d) The numbers in the last column of the table are close to 3, suggesting that $C \approx 3D$. (Actually $C = \pi D$, but students are not expected to know this until the answers to the exercise have been discussed.)
e) The circumference is 31.4 cm
f) The diameter is $25 \div 3.14 \approx 7.96$ cm.

❻ Answers will vary. However, among the most obvious options are:
- make the cans smaller
- keep the volume the same, but make the cans shorter and fatter.
- mix aluminum with some other metal to reduce the amount of aluminum needed.

The scoring guide presented below has been developed using student responses on a field test conducted in 1998. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 1

CONCEPTS	Level 1	Level 2	Level 3	Level 4
Understanding of the Relationship between the Circumference C of a Circle and its Diameter D, and that C/D is the Same for All Circles. (exercise ❺) M 8-7, M 8-8, M 8-9, M 8-11	• The response to Exercise ❺ includes fewer than three of the following elements: • a completed table showing C/D between 3.0 and 3.3. • a formula of the form $C = \pi D$ or $C \approx 3.14D$. • an appropriate graph comparing C and D for several cans. • correct application of the formula relating C and D.	• The response to Exercise ❺ includes at least three of the following elements: • a completed table showing C/D between 3.0 and 3.3. • a formula of the form $C = \pi D$ or $C \approx 3.14D$. • an appropriate graph comparing C and D for several cans. • correct application of the formula relating C and D.	• The response to Exercise ❺ includes all of the following elements: • a completed table showing C/D between 3.0 and 3.3. • a formula of the form $C = \pi D$ or $C \approx 3.14D$. • an appropriate graph comparing C and D for several cans. • correct application of the formula relating C and D.	In addition to Level 3: • The graph comparing C and D for several cans is a scatterplot or line graph.

Achievement Levels Defined by the Ministry of Education and Training

Level 1 Identifies achievement that falls much below the provincial standard.

Level 2 Identifies achievement that approaches the standard.

Level 3 The "provincial standard", identifies a high level of achievement of provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.

Level 4 Identifies achievement that surpasses the standard.

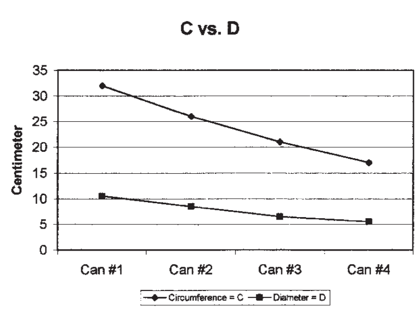
WHAT YOU MIGHT SEE

Level 2

No, I was not surprised because my estimation skills aren't very good and I know that.

b)

	Circumference = C	Diameter = D	C/D
Can #1	32 cm	10.5 cm	3.05 cm
Can #2	26 cm	8.5 cm	3.06 cm
Can #3	21 cm	6.5 cm	3.23 cm
Can #4	17 cm	5.5 cm	3.09 cm



c) The relationship is circumference will always be bigger than diameter because the diameter is always multiplied by pi to find the circumference of a circle.

d) All the quotients are roughly three. So a formula to find the circumference is C = D x 3.

e) 30cm

f) 8.3 cm

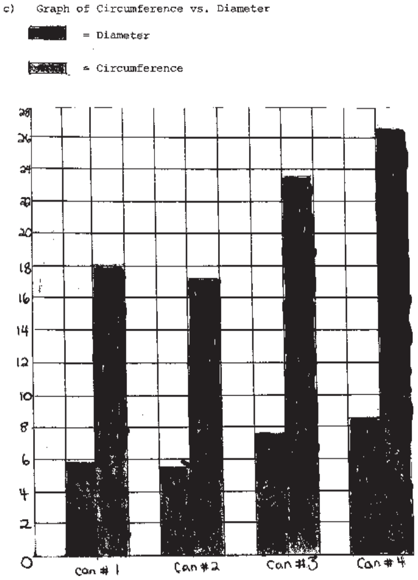
This student has shown a completed table with C/D in the appropriate range. The circumferences and diameters are graphed for four cans and displayed on a single graph so that C and D can be compared, however the graph is not C vs. D and is inappropriate for displaying a relationship between C and D. The points on each graph are joined by line segments, suggesting that the variable on the horizontal axis is continuous. The student has used the formula $C = 3D$ rather than the more precise $C = \pi D$ or $C \approx 3.14D$. However the formula has been applied correctly. This work could be classified as Level 2 or Level 3, depending on how much latitude you allow in your assessment of what constitutes an "appropriate graph."

Level 3

b)

Can #	C (cm)	D (cm)	C/D
1	18.2	5.8	3.14
2	17.3	5.5	3.15
3	23.6	7.5	3.15
4	26.7	8.5	3.14

5 c) Graph of Circumference vs. Diameter



5. a) cont. No, I wasn't surprised. I sized up the diameter of each can with my eye and pictured how long approximately three times that diameter would be ($c = \pi d$). I saw that it was longer than the can's actual height, so that means that the circumference on each of my four cans was longer than the height of each can.

My graph suggests that the diameter is always approx. 1/3 of the circumference. This is true, because the formula for finding circumference is this: $C = \pi d$ (C = 3.14 x d)

WHAT YOU MIGHT SEE

Level 3 **cont'd**

5. d) The pattern in the C/D column of my chart is this: all the quotients are approx. 3.14 or π. The formula for finding circumference is $C = \pi d$

e) The circumference of a can with the diameter of 10 cm would be 31.4 cm.

$$C = \pi d$$
$$= 3.14 \times 10$$
$$= 31.4 \text{ cm}$$

f) The diameter of a can with a circumference of 25 cm is 8 cm.

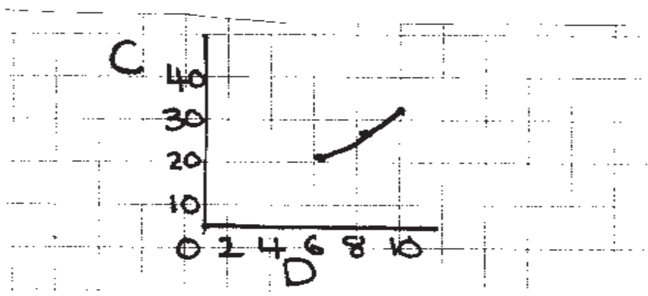
$$d = \frac{c}{\pi}$$
$$= \frac{25}{3.14}$$
$$= 7.95 \text{ (8 cm)}$$

This student has shown a completed table with C/D in the appropriate range. The circumferences and diameters are graphed for four cans and displayed in a double bar graph so that C and D can be compared, although this is not, strictly speaking, a graph of C vs. D. The graph may be regarded as appropriate, albeit not optimal in displaying a linear relationship between C and D. The student has used the appropriate formula $C = \pi D$ and has applied it correctly in the calculation of C and D.

Level 4

	Circumference	Diameter	C/D
Can # 1	20.5cm.	6.5cm.	3.15
Can # 2	27cm.	8.5cm.	3.18
Can # 3	31.5cm.	10cm.	3.15
Can # 4	27cm	8.5cm.	3.18

c)



The graph shows that c/d is always almost pi. Therefore we can use 3.14 (pi) as the base for all calculations relating to diameter and circumference.

d) The pattern in the chart is that all the c/d's are within 0.02 of pi. This would mean that the formula is, pi (3.14) x diameter = circumference. Also circumference divided by pi equals diameter.

e) 10 x 3.14 = 31.4cm

f) 25 divided by 3.14 =7.96 cm.

This work satisfies all the criteria for Level 4 achievement presented in the scoring guide on page 65. In particular the graph of C vs. D shows some understanding of the significance of a scatterplot in identifying relationships between two variables. (Two of the cans that the student chose had the same diameter, so the graph has only three points.)

Activity 2 – Teacher Edition

The 3-Step Process for Making Beverage Cans

Expectations Addressed

M 8-6 measure the radius, diameter, and circumference of a circle using concrete materials.

M 8-8 develop the formula for finding the circumference and the formula for finding the area of a circle.

M 8-9 estimate and calculate the radius, diameter, circumference, and area of a circle, using a formula in a problem-solving context.

M 8-10 draw a circle given its area and/or circumference.

Context

This activity presents the 3-step process in the manufacture of beverage cans. It develops the formula for the area of the circle in terms of its diameter in the context of calculating the wastage when the circular disk of aluminum is stamped out of a rectangular sheet.

In the culmination of Activity 1, students were asked, how engineers might design aluminum cans so that they use less aluminum. While there are many possibilities, this unit investigates three measures to reduce the amount of new aluminum needed, including two ways related to manufacturing:

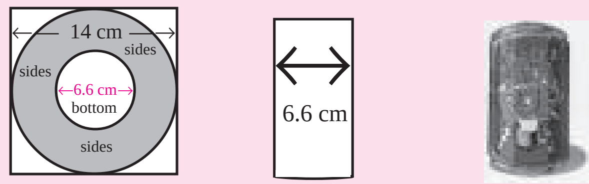
- reducing wastage when circular disks are punched out of rectangular sheets.
- changing the diameter:height ratio of the aluminum can.
- recycling the aluminum cans.

Activity 2 focusses on the first of these three measures. The first step in the construction of aluminum beverage cans is to punch out circular disks of aluminum with diameter 14 cm. The wastage of aluminum in this step are the scraps between the circumference of each circle and the square in which the circle is inscribed. Calculating the area of this wastage involves calculating the area of the circle and then subtracting this area from the area of the square.

Activity 2 – Student Page

The 3-Step Process For Making Beverage Cans

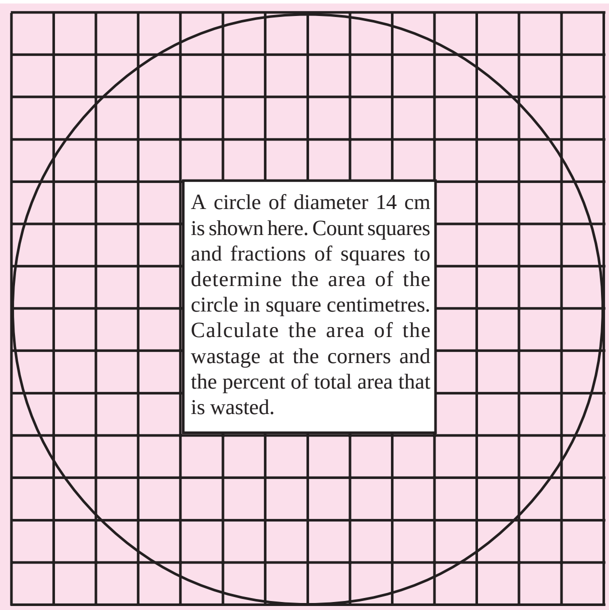
Step 1 → Step 2 → Step 3



A circular disk of diameter 14 cm is cut out of a sheet of aluminum alloy.	The circular disk is stretched into a cylinder of diameter 6.6 cm and height of about 13 cm.	The top of the cylinder is trimmed and tapered to allow for a smaller lid, and then cleaned and decorated.

The activity that follows explores the waste of aluminum in step 1 of the process.

What is the Percentage Wastage When a Circle is cut from a Square?



A circle of diameter 14 cm is shown here. Count squares and fractions of squares to determine the area of the circle in square centimetres. Calculate the area of the wastage at the corners and the percent of total area that is wasted.

To develop the formula for the area of a circle in terms of its diameter, students trace the circumferences of four cans onto centimetre paper. The area of each can is estimated by counting squares and fractions of squares. These areas and the corresponding diameters are entered into a table or spreadsheet and the ratio A/D^2 is calculated for each can. A graph of A vs. D^2 is also constructed by the students who are led to the discovery that A/D^2 is a constant. The formula $A = (\pi/4)D^2$ is then formalized and applied to the calculation of A for a given value of D, and the calculation of D for a given value of A.

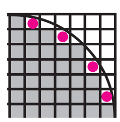
The Lesson Launch 5 minutes

The problem addressed in this Activity is the two-dimensional case of a more mathematical problem that mathematicians call the *Packing of Spheres* (see Sloane reference on p. 96). To motivate the investigation of the relationship between the diameter of a circle and its area, distribute a copy of page 70 to each student. Describe the three-step process for constructing the beverage can as described on that page. Talk about the wastage involved in cutting circular disks from rectangular sheets of aluminum alloy. Display an overhead transparency of the circle on page 70 and pose questions such as:

- *What percentage of this square sheet of aluminum would you estimate would be wasted if this circular disk were punched out of it?*
- *If the square sheet and the circular disk were doubled in size would the percentage wastage be greater?*
- *How might you calculate the percentage of wastage?*

Paired Activity 30 minutes

Group students in pairs and provide each pair with four cans of different diameters, a tape measure calibrated in millimetres, a copy of page 71, and a sheet of centimetre paper (see template p. 56). Have students complete Exercises ❶– ❹ on page 71. In Exercise ❶, students are required to count the number of centimetre squares inside the circle on page 70. One of the easiest ways to do this is to use the symmetry of the circle and count the squares in a quarter sector as shown here.



First shade in the 33 squares that are almost completely contained within the circle. Then draw coloured circles in the 4 squares that are about 3/4 inside the circle. Then assign an area of 1/2 to the remaining 4 squares that are mostly outside the circle. The total area inside a quadrant of the circle is therefore about 33 + 4(3/4) + 4(1/2) or 38 cm^2. Hence the total area inside the circle is about 4×38 or 152 cm^2. A similar procedure in Exercise ❷ should lead students to a total area of about 200 small squares or about 12.6 cm^2. In both cases the wastage should be about $22 \pm 8\%$. As you circulate around the room, check that the numbers in the A/D^2 column of the students' tables are close to $\pi/4$ or 0.79. Check also that the points on the graph lie close to the line of slope $\pi/4$ that passes through the origin.

Activity 2 – Student Page

To do these exercises, you will need four empty cans (including a beverage can) of different diameters, a measuring tape marked in millimetres, a sheet of squared centimeter paper, a pair of scissors, and compasses.

❶ This exercise refers to the grid and circle shown on page 70.

a) The area of the 14×14 square grid is _____________ cm^2.

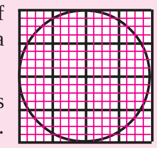
b) The area of the circle of diameter 14 cm is _____________ cm^2.

c) The total area wasted at the corners is ____________ cm^2.

d) The area of the circle as a fraction of the area of the square is ______.

e) The area wasted at the corners as a fraction of the area of the square is ___.

f) The area wasted at the corners as a percent of the area of the square is ____.



❷a) Count squares in the diagram to determine the percent of wastage when a circle of diameter 4 cm is cut out of a grid.

b) Is the percentage of material wasted in ❷ a) the same as the percentage wasted in ❶ f)? Explain why or why not.

❸ The manufacturers of beverage tins arrange circles on long sheets of aluminum alloy so that there is a minimum wastage. Using compasses or your can as a template, draw circles of diameter 6.6 cm on your squared centimetre paper, so that there is smallest possible wastage. Calculate the percentage wastage if your pattern were continued on long sheets. Were you able to achieve a smaller wastage than in Exercise ❷? Explain why or why not.

❹a) Trace the circular top or bottom of a can onto centimetre paper. Count squares to approximate its area. Record its diameter D and its area A. Repeat this for the other cans.

b) Enter in a spreadsheet or in a table like this your measures of the diameter and area of each can. Calculate A/D^2 for each can and enter its value in your table.

	Area of Top A	Diameter D	A/D^2
CAN #1			
CAN #2			
CAN #3			
CAN #4			

c) Make a graph of A vs. D^2 on your centimetre paper or using the *graph* or *chart* menu on your spreadsheet. What does your graph suggest about the relationship between the area of the top of a cylinder and the square of its diameter? Explain.

d) Describe any pattern in the numbers in the A/D^2 column of your table. Use this pattern to write a formula for A in terms of D^2.

e) Use your formula to calculate the area of the top of a can of diameter 10 cm.

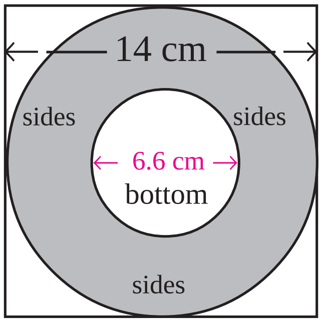
f) Use your formula to calculate the diameter of a can with a top of area 50 cm^2.

Closure

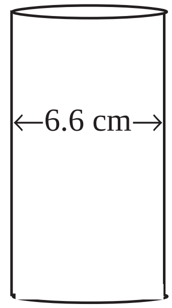
When all the students have finished, ask for volunteers to share their findings with the class. Ask, "Does the percentage wastage depend on the size of the circle and square? What is the relationship between the area of a circle and the square of its diameter? By what factor must we multiply the square of its diameter to obtain the area inside a circle"?Then call out diameters of various circles and ask them to estimate, using their calculators, the corresponding areas. Then call out the areas of different circles and ask them to estimate the corresponding diameters. Ensure that they know how to calculate square roots on their calculators.

The 3-Step Process for Making Beverage Cans

Step 1 → **Step 2** → **Step 3**



A circular disk of diameter 14 cm is cut out of a sheet of aluminum alloy. A punch of diameter 6.6 cm pushes downward, creating the sides of the can.

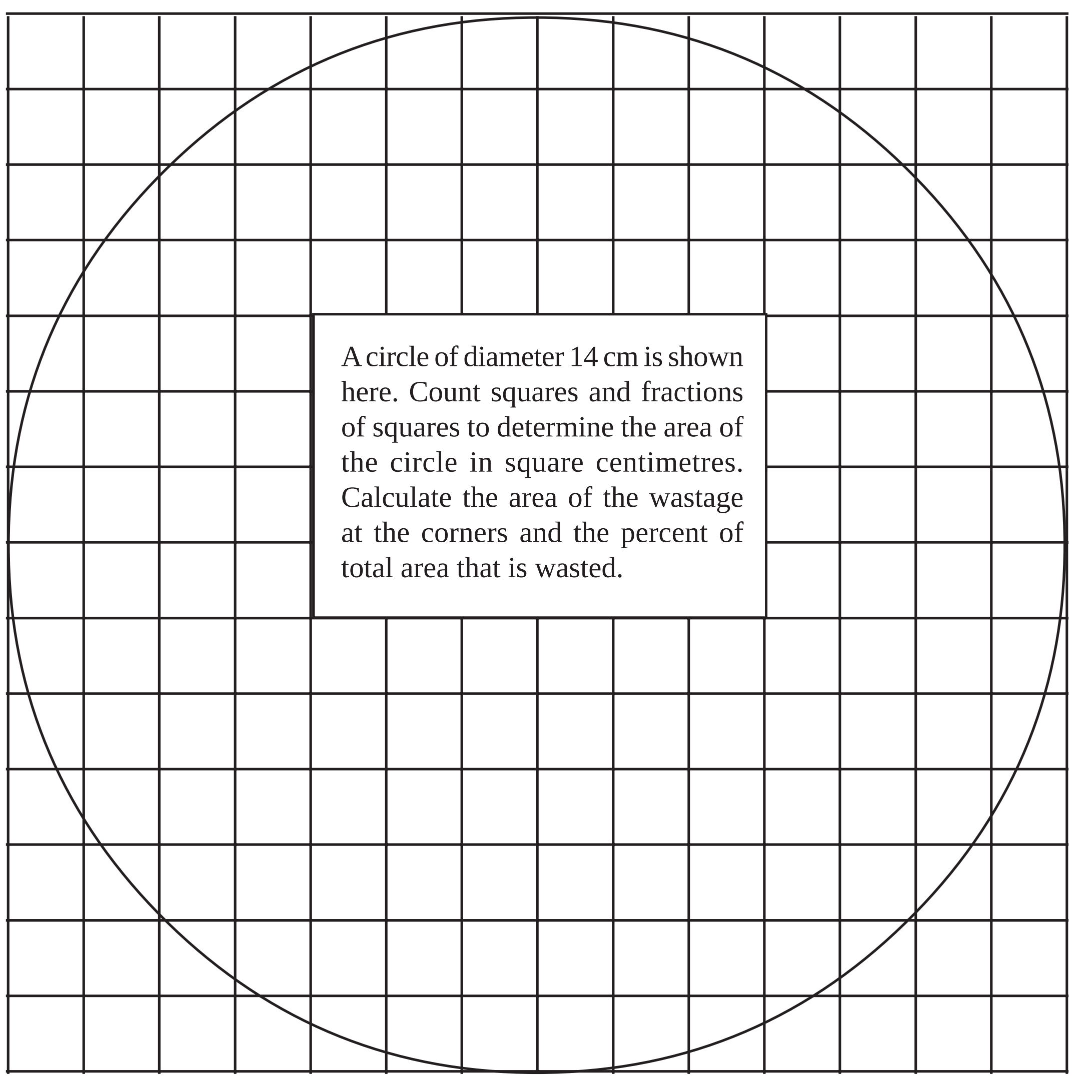


The circular disk is stretched into a cylinder of diameter 6.6 cm and height of about 13 cm.

The top of the cylinder is trimmed and tapered to allow for a smaller lid, and then cleaned and decorated.

The activity that follows explores the waste of aluminum in step 1 of the process.

What is the Percentage Wastage When a Circle is cut from a Square?



A circle of diameter 14 cm is shown here. Count squares and fractions of squares to determine the area of the circle in square centimetres. Calculate the area of the wastage at the corners and the percent of total area that is wasted.

To do these exercises, you will need four empty cans (including a beverage can) of different diameters, a measuring tape marked in millimetres, a sheet of squared centimeter paper, a pair of scissors, and compasses.

❶ This exercise refers to the grid and circle shown on page 70.

a) The area of the 14×14 square grid is ______________ cm^2.

b) The area of the circle of diameter 14 cm is about ______________ cm^2.

c) The total area wasted at the corners is about _____________ cm^2.

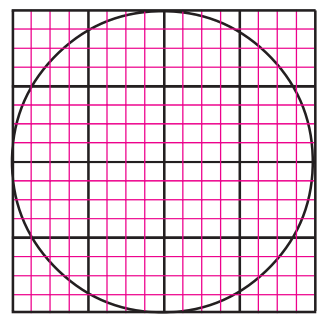
d) The area of the circle as a fraction of the area of the square is about _________

e) The area wasted at the corners as a fraction of the area of the square is about _________

f) The area wasted at the corners as a percent of the area of the square is about _________

❷ a) Count squares in the diagram to determine the percent of wastage when a circle of diameter 4 cm is cut out of a grid.

b) Is the percentage of material wasted in ❷ a) the same as the percentage wasted in ❶ f)? Explain why or why not.



❸ The manufacturers of beverage tins arrange circles on long sheets of aluminum alloy so that there is a minimum wastage. Using compasses or your can as a template, draw circles of diameter 6.6 cm on your squared centimetre paper, so that there is smallest possible wastage. Calculate the percentage wastage if your pattern were continued on long sheets. Were you able to achieve a smaller wastage than in Exercise ❷? Explain why or why not.

❹ a) Trace the circular top or bottom of a can onto centimetre paper. Count squares to approximate its area. Record its diameter D and its area A. Repeat this for the other cans.

b) Enter in a spreadsheet, or in a table like this, your measures of the diameter and area of each can. Calculate A/D^2 for each can and enter its value in your table.

	Area of Top A	**Diameter D**	**A/D^2**
CAN #1			
CAN #2			
CAN #3			
CAN #4			

c) Make a graph of A vs. D^2 on your centimetre paper or using the *graph* or *chart* menu on your spreadsheet. What does your graph suggest about the relationship between the area of the top of a cylinder and the square of its diameter? Explain.

d) Describe any pattern in the numbers in the A/D^2 column of your table. Use this pattern to write a formula for A in terms of D^2.

e) Use your formula to calculate the area of the top of a can of diameter 10 cm.

f) Use your formula to calculate the diameter of a can with a top of area 50 cm.

Grade 8
Answer Key for Activity 2

❶ a) 14^2 or 196 cm^2.

b) The area of a circle of diameter 14 cm is $\pi \times 7^2$ or about 154 cm^2.

c) The total area wasted at the corners is about 196 – 154 or about 42 cm^2.

d) The area of the circle as a fraction of the area of the square that contains it is about 154/196 or about 0.786

e) The area wasted at the corners as a fraction of the area of the square is about $1 - 0.786 = 0.214$.

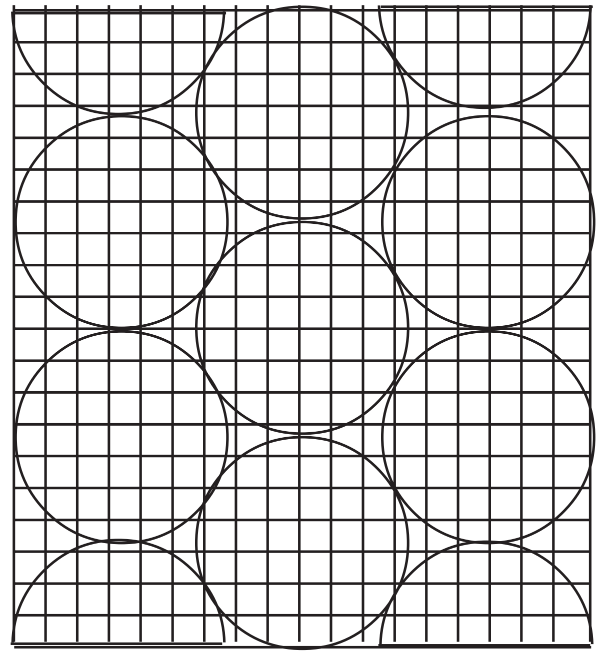
f) The area wasted at the corners as a percent of the area of the square is about 21.4%.

❷ a) The answer is the same as ❶ f), i.e., 21.4%. Students should have answers between 14% and 30%.

b) The percentage of material wasted is the same as in ❶ f) because merely enlarging or shrinking the circle and the grid together does not change the proportion of the area taken by the circle.

❸ Answers will vary depending upon how many circles the student draws and how they are arranged. It is expected that students will draw circles and count the squares outside the circles to obtain the percentage wastage. By using symmetry, students need only count the squares bounded by any 3 circles and this applies to all such regions. Similarly the number of squares bounded by two circles and a border need only be counted once.

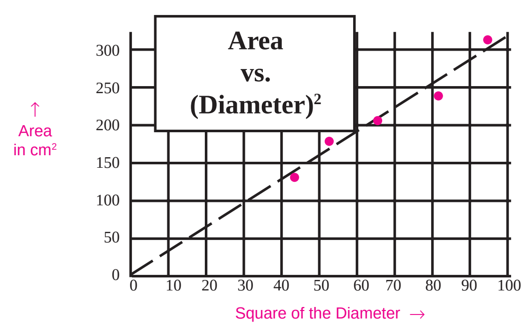
The optimum arrangement with up to 7 full circles is the *hexagonal lattice packing* shown in the diagram above. This yields 14% wastage. The proof of this (not to be shared with the students) is as follows. A single circle is surrounded by 6 circles whose centres are the vertices of a hexagon. Each of the 7 circles has an area of πR^2 square units and is cut out of a rectangular grid of height $6R$ and width $2(\sqrt{3} + 1)R$. The total area of the 7 circles as a fraction of the area of the grid is therefore $7\pi/[12(\sqrt{3} + 1)]$, i.e., 0.67 or about two-thirds. The wastage would be about one-third or about 0.33. However, if we consider a long sheet, we can insert four semi-circles to fill the remainder of grid and repeat this pattern so that each semi-circle becomes a circle. The area of the 7 circles and 4 semi-circles, i.e., 9 circles is $9\pi/[12(\sqrt{3} + 1)]$, i. e., 0.86 or 86%, leaving wastage of about 14%.



❹ a) Answers will vary.

b) $A/D^2 = \pi/4$ for all cans. Student measurements should generate values of A/D^2 somewhere between 0.65 and 0.85.

c) The graph should look something like this.



The points on the graph lie close to a straight line. This suggests that area is a linear function of the square of the diameter.

d) The numbers in the last column of the table are close to 0.75 suggesting that $A \approx 3D^2/4$. (Actually $A = \pi D^2/4$, but students are not expected to know this prior to completing this investigation.)

e) The area is $25\pi \approx 78.54$ cm^2.

f) The diameter is $2\sqrt{50 \div \pi} \approx 7.98$ cm.

The scoring guide presented below has been developed using student responses on a field test conducted in 1998. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 2

	Level 1	Level 2	Level 3	Level 4
PROBLEM SOLVING **Selection of an Appropriate Strategy for Cutting Circles from a Rectangle With Minimum Wastage.** (exercise ❸) M 8-4, M 8-9	• Little or no evidence of understanding that the area of a circle inscribed in a square is a fixed fraction of the area of the square and is independent of its diameter. • Is unable to arrange circles so that the wastage is less than 21.5%.	• Evidence of understanding that the area of a circle inscribed in a square is a fixed fraction of the area of the square and is independent of its diameter. • Is unable to arrange circles so that the wastage is less than 21.5%.	• Evidence of understanding that the area of a circle inscribed in a square is a fixed fraction of the area of the square and is independent of its diameter. • Is able to arrange circles so that the wastage is less than 21.5%.	In addition to Level 3: • Is able to arrange the circles so the wastage is less than 15%. • Displays computation to verify that the wastage is less than 15%.
CONCEPTS **Understanding of the Relationship between the Area of a Circle and its Diameter.** (exercise ❹) M 8-4, M 8-8, M 8-9	• The response to Exercise ❹ includes fewer than three of the following elements: • a completed table showing A/D^2 between 0.65 and 0.85. • a formula of the form $A = \pi D^2/4$ or $A \approx 0.79D^2$. • an appropriate graph comparing A and D^2 for several cans. • correct application of the formula relating A and D^2.	• The response to Exercise ❹ includes at least three of the following elements: • a completed table showing A/D^2 between 0.65 and 0.85. • a formula of the form $A = \pi D^2/4$ or $A \approx 0.79D^2$. • an appropriate graph comparing A and D^2 for several cans. • correct application of the formula relating A and D^2.	• The response to Exercise ❹ includes all of the following elements: • a completed table showing A/D^2 between 0.65 and 0.85. • a formula of the form $A = \pi D^2/4$ or $A \approx 0.79D^2$. • an appropriate graph comparing A and D^2 for several cans. • correct application of the formula relating A and D^2.	In addition to Level 3: • The graph comparing A and D^2 for several cans is a scatterplot or line graph.

WHAT YOU MIGHT SEE

Problem Solving: Devising A Strategy for Packing Circles with Minimum Wastage

Level 4

2. The square's area is 256 $\begin{array}{r}16\\ \times 16\\ \hline 256\end{array}$
There is 13 squares wasted at each corner. $\begin{array}{r}13\\ \times 4\\ \hline 52\end{array}$ The total wastage is 51.

$\frac{52}{256} = \frac{26}{128}$ 128)26 20%

The wastage in percent is 20% wastage.

b) I don't think that the percent of material waste changes if the diameter of the circle changes. If the size of the circle changes so does the size of the square. Then if the size of the square changes so does the amount of wastage. If you change the amount of squares in the cirche by a certain amount then you change the amount of wastage by the same amount. This means the percent says around the same.

3.

$A = \pi r^2$
$A = 3.14 \times 3.3 \times 3.3$
$A = 3.14 \times 10.89$
$A = 34.2$

Area of one circle

$\begin{array}{r}34.2\\ \times 9\\ \hline 307.75\end{array}$ $\begin{array}{r}356.4\\ -307.75\\ \hline 48.65\end{array}$

Area of sheet $= 18 \times 19.8 = 356.4$

$\frac{48.65}{356.4} = 14\%$

Yes I was able to achive less wastage because insted of giving each circle a whole square piece of aluminum around it, I fit them in closer together. I filled in the gaps. This means I fit more circles per area and had less wastage. I had 14% wastage insted of a 20% wastage.

The student used symmetry to estimate a wastage of 13 cm² at each corner and multiplied by 4 to obtain a wastage of 52 cm² out of a total area of 256 cm². This yielded an estimated wastage of about 20% which is close to the exact amount. The student states, "I don't think the percent of material waste changes if the diameter of the circle changes." In response to Exercise ❸, the student drew seven circles in three rows so the rows were staggered (as on p. 72) and then measured the width and calculated the height of the rectangle containing them. The student then displayed a calculation showing that this arrangement of circles yields a minimal wastage of 14%.

WHAT YOU MIGHT SEE

Concepts: Understanding the Relationship between the Area of a Circle and its Diameter

Level 1

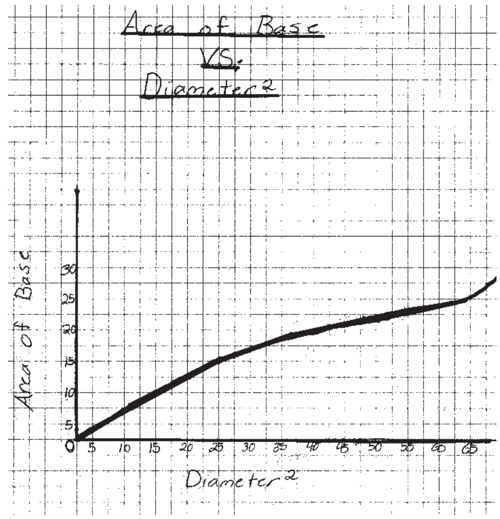
4a.) Can 1- d= 8cm
A= 3.14 x 8 = 25.12cm²

Can 2- d= 5cm
A= 3.14 x 5 = 15.7cm²

Can 3- d= 6cm
A= 3.14 x 6= 18.84cm²

Can 4- d= 7cm
A= 3.14 x 7= 21.98cm²

	Area of Base A	Diameter D	A/D2
Can 1	25.12cm²	8cm	0.39
Can 2	15.7cm²	5cm	0.63
Can 3	18.84cm²	6cm	0.52
Can 4	21.98cm²	7cm	0.45



c.) It suggests that there is not much of a gap between area of base and diameter squared. That they are fairly close together and the line in the graph doesn't rise very steeply.

d.) The area divided by the diameter squared starts when the area is 1 the A/D2 is 9.09 and starts going down when every area of base number increases. When you get to ten it starts to go into just decimals. For ten it is .90 and goes downward from there. A formula would be Diameter multiplied by two and then area divided by the diameter squared.

e.) A= 10cm D= 3.18
3.18 x 3.18 = 10.11
10÷10.11= .99
The A/D2 is .99

f.)A= 50cm D= 15.92
15.92 x 15.92 = 253.45cm
50÷253.45= .2
The A/D2 is .2

The values of A/D^2 in the table are outside reasonable limits because the student, instead of counting squares, calculated the areas of the bases of the cans and used the incorrect formula A = 3.14D. The incorrect data generated incorrect values for A/D^2, and so the graph was not linear and the student could not discover an appropriate formula. With this compounding of errors, the student was unable to apply an appropriate formula to calculate A or D correctly

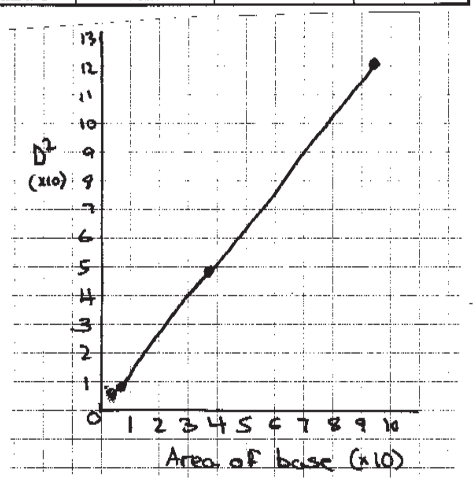
Level 4

4. a) Can # 1 a→40cm². d→7cm.
Can # 2 a→93cm². d→11cm.
Can # 3 a→7cm². d→3cm.

b)

	Area of Base A	Diameter D	A/D^2
Can # 1	38.5cm²	7	0.79
Can # 2	95cm²	11	0.79
Can # 3	7cm²	3	0.78

c)



d) The numbers in the A/D^2 column of my table are all about 0.79. The formula is 0.79 x d². To find the diameter it is A/0.79 then find the square root of that answer.
e) 10 x 10 x 0.79 = 79 cm²
f) 50/0.79 = 63.29 →7.955

This work satisfies all the criteria for Level 4 achievement presented in the second row of the scoring guide on page 73. The values of A/D^2 in the table are close to the true value $\pi/4 \approx 0.785$ The graph of D^2 vs. A shows some understanding of the significance of a scatterplot in identifying relationships between two variables. (Actually, Exercise ❹ c asked for the graph of A vs. D^2, but at this stage we're delighted when students select the most appropriate kind of graph.)

This student also presents a correct formula for the area in terms of diameter ($A = 0.79\ D^2$) and applies it correctly in the calculation of both D and A.

Activity 3 – Teacher Edition
Which Can Is Most Economical?

Expectations Addressed

M 8-2 explain the relationships between various units of measurement.

M 8-4 make increasingly more informed and accurate measurement estimations based on an understanding of formulas and the results of investigations.

M 8-6 measure the radius, diameter, and circumference of a circle using concrete materials.

M 8-9 estimate and calculate the radius, diameter, circumference, and area of a circle, using a formula in a problem-solving context.

Context

In Activity 2, students investigated the first of these three ways of reducing the amount of new aluminum required for the manufacture of beverage cans:

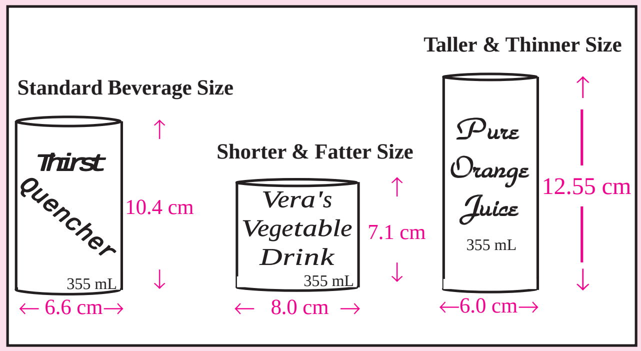
- reducing wastage when circular disks are punched out of rectangular sheets.
- changing the diameter:height ratio of the aluminum can.
- recycling the aluminum cans.

They learned that there is about 14% wastage when circular disks are punched out of aluminum sheets. Hence, reusing the scraps could reduce the new aluminum needed by no more than 14%. In Activity 3 students investigate the second of these three ways of reducing aluminum consumption. By applying the formulas for circumference and area of circles they calculate the surface areas of cylindrical cans of various circumference:height ratios. In this way they determine that the circumference to height ratio of the standard beverage can is not optimal. In fact, of all cylindrical cans that hold 355 mL of beverage, the cylinder with a height to diameter ratio of 1:1 has the smallest surface area. Students are invited to suggest why the most economical shape may not be desirable.

Activity 3 – Student Page

Which Can Is Most Economical?

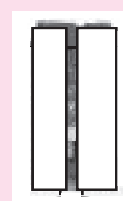
These cans all hold about 355 mL of beverage. However, they have different surface areas and therefore contain different amounts of metal. Which of them has the smallest surface area (and therefore uses the least amount of metal)?



Calculating the Surface Area of the Curved Surface of a Can

To do this exercise, you will need an empty beverage can, a measuring tape marked in millimetres, a pair of scissors, and a sheet of centimetre paper.

❶ a) Cut a rectangle from a notebook-size sheet of paper so that it wraps snugly around the curved surface of your beverage can. (Include the neck and the tapered part at the bottom.)Trim the rectangle so that both ends touch without overlapping. Tape the ends together to hold the rectangle in place while you trim off any part that extends above or below the curved surface.



b) Cut the tape and flatten your paper. Calculate the area of the rectangle in square centimetres. This is the surface area of the curved surface of your can.

c) The curved surface of a beverage can is pressed to a thickness of 0.08 mm. Calculate the volume (in cubic centimetres) of aluminum in the curved surface of a beverage can.

In Exercise ❶, students discover by wrapping a sheet of centimetre paper around a beverage can that the curved surface of a cylinder, when rolled flat, is a rectangle of length C and width *h*, where C is the circumference of the cylinder and *h* is its height. By calculating the area of this rectangle, they obtain the surface area of the curved surface of a cylinder. Exercise ❷ extends this activity by calculating the areas of the top and bottom of a cyliner to obtain its total surface area. In Exercise ❸, students use the formulas $C = \pi D$ and $A = \pi D^2/4$ to calculate the circumferences and surface areas of three different cylindrical cans. By recording these surface areas in a table, they discover that the standard can is not optimal. Finally in Exercise ❹, students are asked to determine which of three different shapes has the largest increase in surface area if the thickness of its top were increased.

The Lesson Launch 5 minutes

Hold up a cardboard cylinder such as the roll from a paper towel or toilet roll dispenser. Pose questions such as the following:

- *How could you determine the area of cardboard needed to make this roll?*
- *What shape do you think it would have if I cut it along the seam and opened it flat?*
- *Is more than one kind of shape possible? What determines the shape?*

Then cut the cardboard roll along its seam and show students that it unrolls into a parallelogram. Ask how we might have determined the length of this parallelogram without cutting the roll. Conduct a discussion leading students to the discovery that the length of the parallelogram is the same as the circumference of the cylinder for which it was the template. Then ask whether the cylinder could have been cut to yield a rectangle. Ask what the dimensions of that rectangle would be. Before moving students into the next activity, ensure they understand that the surface area of a cylinder is defined to be the area of any template from which it is made.

Paired Activity 30 minutes

Group students in pairs and provide each pair with a beverage can, a tape measure calibrated in millimetres, a copy of pages 78 and 79, a pair of scissors, and a sheet of centimetre paper (see template p. 56). Have students complete Exercises ❶–❸. Circulate around the classroom as students complete Exercise ❶ to ensure that students are including the neck and the tapered bottom. The area of the curved surface they obtain should be between 200 and 230 cm^2. Some students may observe that the paper does not fit snugly around the tapered neck and so the area calculation will not be accurate. Stress that this procedure will yield only an approximation, and ask the students whether the tapered neck makes the surface area larger or smaller than an untapered neck.

As the students move toward the completion of Exercise ❸, check that their calculations of total surface area recorded in their tables are the right order of magnitude.

Activity 3 – Student Page

❷ a) In Exercise ❶, you calculated the area of the curved surface of your beverage can. To find the total surface area of your beverage can you will need to calculate the areas of the top and bottom of the can. Use the formula you discovered in Activity 2 for the area of a circle to calculate the area of the top and bottom.

b) What is the total surface area of your beverage can?

❸ a) In Exercise ❶, you may have discovered that the surface area of the curved surface of a cylinder is the product of the circumference and the height. Use this information to calculate the surface area of the curved surfaces of the three cans shown on page 78.

b) Calculate the surface areas of the top and bottom of each of the cans shown on page 78. Record your answers in this table.

	Area of Curved Surface	Area of the top	Area of the bottom	Total Surface Area
Standard Beverage Can				
Shorter & Fatter Can				
Taller & Thinner Can				

c) Calculate the surface area of each can and record it in the table. Which can is the most economical? Give reasons for your answer.

d) Is the standard size beverage can the most economical? Why might a soft drink company use a can which does not have the most economical dimensions?

❹ For extra strength, the top of the beverage can is made thicker than the sides. Which can would show the largest increase in the amount of aluminum used? Give reasons for your answer.

CHALLENGE

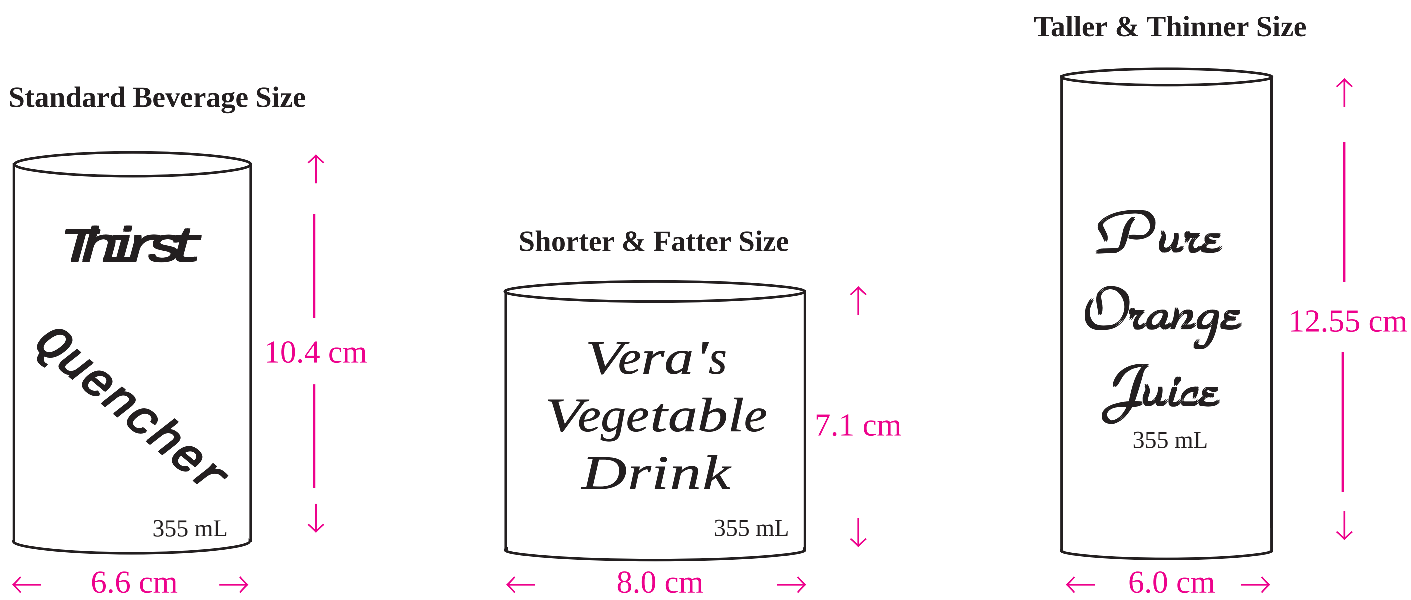
If D denotes the diameter of a can, and h denotes its height, can you write an expression for the area of the curved surface of the can in terms of D and h? Show your work.

Closure

When the students have completed Exercise ❸, ask one pair to display its table on the blackboard or overhead. Invite other students to contribute answers that differ from those displayed. If there are some differences of opinion and distinctly different answers, work through the computations with the class to demonstrate the correct computation of the surface areas. When students have shown that they understand how to compute the surface area of a cylinder, and recognize that the short, fat cylinder has smaller surface area than the tall thin cylinder, ask why the short fat cylinder is not used as a beverage can. Discuss the esthetics, the comfort of gripping a smaller can, and the psychological illusion that there is more beverage in a taller can. Bring the lesson to a close with a brief discussion of how the amount of material in a can of each shape is affected by an increased thickness in the top.

WHICH CAN IS MOST ECONOMICAL?

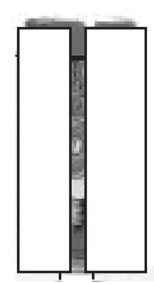
These cans all hold about 355 mL of beverage. However, they have different surface areas and therefore contain different amounts of metal. Which of them has the smallest surface area (and therefore uses the least amount of metal)?



CALCULATING THE SURFACE AREA OF THE CURVED SURFACE OF A CAN

To do this exercise, you will need an empty beverage can, a measuring tape marked in millimetres, a pair of scissors, and a sheet of centimetre paper.

❶ a) Cut a rectangle from a notebook-size sheet of paper so that it wraps snugly around the curved surface of your beverage can. (Include the neck and the tapered part at the bottom.) Trim the rectangle so that both ends touch without overlapping. Tape the ends together to hold the rectangle in place while you trim off any part that extends above or below the curved surface.



b) Cut the tape and flatten your paper. Calculate the area of the rectangle in square centimetres. This is the surface area of the curved surface of your can.

c) The curved surface of a beverage can is pressed to a thickness of 0.08 mm. Calculate the volume (in cubic centimetres) of aluminum in the curved surface of a beverage can.

❷ a) In Exercise ❶, you calculated the area of the curved surface of your beverage can. To find the total surface area of your beverage can you will need to calculate the areas of the top and bottom of the can. Use the formula you discovered in Activity 2 for the area of a circle to calculate the area of the top and bottom.

b) What is the total surface area of your beverage can?

❸ a) In Exercise ❶, you may have discovered that the surface area of the curved surface of a cylinder is the product of the circumference and the height. Use this information to calculate the surface area of the curved surfaces of the three cans shown on the other sheet.

b) Calculate the surface areas of the top and bottom of each of the cans shown on the other sheet. Record your answers in this table.

	Area of Curved Surface	**Area of the top**	**Area of the bottom**	**Total Surface Area**
Standard Beverage Can				
Shorter & Fatter Can				
Taller & Thinner Can				

c) Calculate the surface area of each can and record it in the table. Which can is the most economical? Give reasons for your answer.

d) Is the standard size beverage can the most economical? Why might a soft drink company use a can which does not have the most economical dimensions?

❹ For extra strength, the top of the beverage can is made thicker than the sides. Which can would show the largest increase in the amount of aluminum used? Give reasons for your answer.

CHALLENGE

If D denotes the diameter of a can, and *h* denotes its height, can you write an expression for the area of the curved surface of the can in terms of D and *h*? Show your work.

GRADE 8

ANSWER KEY FOR ACTIVITY 3

❶ b) For a standard 355-mL beverage can, the area of the curved surface is $6.6\pi \times 10.4$ cm^2 or 215.6 cm^2.
c) The thickness of the aluminum is 0.008 cm, and so the volume is 215.6×0.008 or 1.72 cm^3.

❷ Answers will vary depending upon the can the students used. The computations for a standard soft drink beverage can are given here.
a) Area of the top of the can is $\pi \times 3.3^2 = 34.2$ cm^2.
b) Total surface area of the can is:
Area of the curved surface + Area of top + Area of bottom
= 215.6 + 34.2 + 34.2 or 284 cm^2.

❸ a) & b) The completed table is given below.

	Area of Curved Surface	Area of the Top	Area of the Bottom	Total Surface Area
Standard Beverage Can	215.6	34.2	34.2	284.0
Shorter & Fatter Can	178.4	50.3	50.3	279
Taller & Thinner Can	236.6	28.3	28.3	293.2

c) The shorter and fatter can is the most economical because all cans have the same capacity (355 mL), but the shorter and fatter can has the smallest surface area and therefore uses the least amount of aluminum. It is easily shown using elementary calculus or a spreadsheet, that of all cylindrical shapes of the same volume, the one with height equal to its diameter has the minimal surface area.

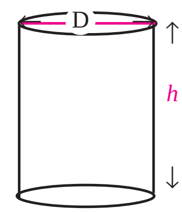
d) The standard size beverage can is less economical in aluminum content than the shorter and fatter can of the same capacity. Soft drink manufacturers use a thinner can because it is easier to hold. A taller and thinner can also appears to hold more soft drink than a can that is shorter and fatter.

❹ The shorter and fatter can has the largest top and bottom area. Any increase in the thickness of the top or bottom of this can would increase the amount of aluminum used in this can more than it would increase the amount of aluminum used in the others.

CHALLENGE

❺ Students may or may not have learned algebraic notation at this time, so this exercise is offered as a challenge to those who have some familiarity with algebra.

When the curved surface is flattened into a rectangle, its length is the circumference πD.
The width or height of the rectangle is h. The area is the product πDh.



TEACHER NOTE:

In this section, the answers are often expressed using π. This is for your information as a teacher. If students have not been using the symbol π prior to the study of this unit, it is sufficient to have them use 3.14 instead. Many students are confused by the formalism of a symbol that represents a particular number and this impedes their understanding of the relationships involved. It is usually best to begin estimation of circumference and area using $C \approx 3D$ and $A \approx 3D^2/4$ (or $A \approx 3R^2$), and then move to $C \approx 3.14D$ and $A \approx 3.14R^2$. When students are ready for the next level of formality, you can introduce the symbol π to replace the symbol $\approx$ with the symbol = in the area formulas.

Note also that it does not matter whether you use the area formula $A = \pi D^2/4$ or the more familiar formula $A = \pi R^2$. Each has its own advantages. We chose the former so that the student investigation in Exercise ❹ of Activity 2 would not be undermined by any prior exposure to the area formula, and to develop the “perceptual anchor”: *the area of a circle is about 3/4 the area of the area of the square in which it is inscribed.*

The scoring guide presented below has been developed using student responses on a field test conducted in 1998. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 3

	Level 1	Level 2	Level 3	Level 4
APPLICATION **Correct Application of the Formulas for the Areas of the Curved and Flat Surfaces of a Cylinder and Appropriate Interpretation of the Results** (exercise ❸) M 8-9 **General Expectation:** Solve problems related to the calculation of the radius, diameter, and circumference of a circle.	• makes few correct computations in the calculation of the following: • the area of the top and bottom of a cylinder. • the surface area of the curved surface of a cylinder. • the total surface area of a closed cylinder. • May make inappropriate interpretations of the results of the area computations.	• makes consistent errors in the calculation of one of the following: • the area of the top and bottom of a cylinder. • the surface area of the curved surface of a cylinder. • the total surface area of a closed cylinder. • Makes appropriate interpretations of the results of the area computations.	• calculates correctly with few minor errors the following: • the area of the top and bottom of a cylinder. • the surface area of the curved surface of a cylinder. • the total surface area of a closed cylinder. • Makes appropriate interpretations of the results of the area computations.	• calculates correctly with no errors the following: • the area of the top and bottom of a cylinder. • the surface area of the curved surface of a cylinder. • the total surface area of a closed cylinder. • Makes appropriate interpretations of the results of the area computations.

WHAT YOU MIGHT SEE

Level 1

Activity 3.

3a/b)

	Area of Curved Surface.	Area of the top	Area of the bottom	Total Area of Surface
Standard Beverage Can.	68.64 cm²	20.724 cm²	20.724 cm²	110.088 cm².
Shorter + Fatter Can.	56.8 cm².	25.12 cm²	25.12 cm²	107.04 cm²
Taller + Thinner Can	75.3 cm².	18.84 cm²	18.84 cm²	112.98 cm²

c) The shorter, fatter can is the most economical, (using 107.04 cm² of tin.) This is because it uses less tin, while holding the same amount as the other two.

d) The standard sized beverage can is not the most economical, but it is the easiest to hold and grip, and it may look better, to the manufacturers.

This student has calculated the surface area of the curved surface of each can by multiplying the diameter by the height. The areas of the top and bottom of each can were obtained by computing πD instead of πR^2 or $\pi D^2/4$. Consequently all the entries in the table in Exercise ❸ b are incorrect.

The student has correctly compared the surface areas in the table and concluded that the shorter and fatter can uses the least amount of "tin."

This student needs to revisit the concepts of area and surface area and learn the relationships and formulas connecting these concepts.

Level 2

2.a)6.5 ÷ 2 =Y

Y² x PI = 33.16625cm²

	Area Of Curved Surface	Area of the top	Area of the bottom	Total Surface Area
Standard Beverage Can	68.64	34.1946	34.1946	137.0292
Shorter & Fatter Can	56.8	50.24	50.24	157.2800
Taller & Thinner Can	75.3	28.26	28.26	131.8200

3.a)See Stapled Sheet

b)See Stapled Sheet

c) The Most econmical would be the tall skinny can because it uses the least aluminum and holds the same amount of liquid as the others.

d) The Soft Drink Company Might Use A can wich is not the most econmicaal beacause ther might be less waste when you cut it out of the square then when you cut out a average can. Another reason is because its easier to carry, or the customers want that size. Also It Might Be Cheaper To Ship In Those Boxes.

4. The Largest Increase Of Aluminum would be on the short and fat can because the surface area of the top is the biggest and since the top and bottom of the can are where the aluminum's thickness is being increased it would affect the can with the largest top and bottom area which in this case is the Short and Fat can.

CHALLENGE. Diameter x 3.14 x height= Curved Surface Area

The table reveals that this student computed correctly the areas of the top and bottoms of the cans. However the student was consistently incorrect in calculating the surface areas of the curved surface because he multiplied diameter by the height. Ironically, the student gives the correct formula in the Challenge problem at the bottom of the page. The student correctly identifies the short and fat can in Exercise ❹.

WHAT YOU MIGHT SEE

Level 3

2) b) a of □ = l x w
= 24 x 9
= 216 cm2

a of ○ = πr2
= 3.14 x 4 x 4
= 50.24cm2

50.24
50.24
+216
316.48 cm2

3) a) standard beverage size

c=πd
=3.14 x 6.6
=20.724 cm2

a of curved =c x h
=20.742 x 10.4
=215.7168 cm2

a of top =πr2
=3.14 x 3.3 x 3.3
= 34.1946 cm2

215.7168
+ 34.1946
249.9114
+ 34.1946
284.106 cm2

shorter and fatter size

c=πd
=3.14x8
=25.12 cm2

a of curved =cxh
=25.12x7.1
=178.352 cm2

a of top =πr2
=3.14x4x4
=50.24

178.352
+50.14
228.492
+ 50.14
278.632

taller thinner size

c=πd
=3.14x6
=18.84 cm2

a of curved =cxh
=18.84x12.55
=236.442 cm2

a of top =πr2
=3.14x3x3
=28.26

236.442
28.26
+ 28.26
348.962

b)

Cm2	Area of curved surface	Area of the top	Area of the bottom	Total surface area
Standard beverage can	215.7168	34.1946	34.1946	284.106
Shorter & fatter can	178.352	50.24	50.24	278.632
Taller & thinner can	236.442	28.26	28.26	348.962

c) I think that the shorter fatter size is the most economical because there is less surface area ∴ there is less material used and saves time and money

d) no the standard can is not the most economical. I think that the soft drink companies don't use the one with the smallest dimensions is because it is more convenient than the others are and easier to hold. It is the right size for the cup holders in the car so if you were to change the size now it will cost a lot of money. It is probably better off to stay with the regular size for now until people find a better way to store liquids.

4) the shorter fatter one because the circumference of the top and bottom is the largest ∴ the more material used on the rim.

All the values in the table have been computed correctly, except for the total surface area of the taller and thinner can. The student computed correctly the areas of the top and bottom and the surface area of the curved surface of the can but added incorrectly.

Appropriate interpretations of the computed results are evident in the answers to Exercises ❸ c & d. The reference to cup holders is particularly delightful.

In the response to Exercise ❹, it is not clear whether the student is aware that it is the increased area more than the increased circumference that is critical.

Level 4

3.a) Thirst Quencher: 215.5 cm²
Vegetable: 178.35 cm²
Orange Juice: 236.55 cm²

b)

	Area of the curved surface	Area of the top	Area of the bottom	Total surface area
Standard beverage can	215.5 cm²	34.2 cm²	34.2 cm²	283.9 cm²
Shorter & fatter can	178.35 cm²	50.25 cm²	50.25 cm²	278.85 cm²
Taller & thinner can	236.55 cm²	28.25 cm²	28.25 cm²	293.05 cm²

c) The shorter and fatter can is most economical because it uses the least amount of aluminum, but you get the same amount of liquid. This will produce less waste and be better for the environment.

d) The standard size is not the most economical but it does use only 5 cm² more. Soft drink company's may use this can because it looks the best, is the easiest to drink from, and is easiest to print designs on. The short and fat can would be to hard to drink from, and the tall one uses to much aluminum.

4. This gives the short and fat can the largest increase because it has the largest diameter and top and bottom surface area. this may make the standard size the most economical. This would then be another reason that the companies use that size.

There are no errors in the values given in the table. The student was able to apply the formulas for the area of a circle and surface area of a cylinder appropriately and to perform the computations without error.

The interpretations of the results were appropriate, coherent and well-expressed. In the response to Exercise ❹, the student suggests yet another reason why manufacturers might have chosen the standard size.

Activity 4 – Teacher Edition
Re-design, Reuse, or Recycle?

Expectations Addressed

M 8-2 explain the relationships between various units of measurement.

M 8-3 research, describe, and report on uses of measurement in projects at home, in the workplace, and in the community that require precise measurements.

M 8-9 estimate and calculate the radius, diameter, circumference, and area of a circle, using a formula in a problem-solving context.

Context

In Activities 2 and 3, students investigated the first two of these three ways of reducing the amount of aluminum required for the manufacture of beverage cans:

- reducing wastage when circular disks are punched out of rectangular sheets.
- changing the diameter:height ratio of the aluminum can.
- recycling the aluminum cans.

In this activity, students investigate the amount of aluminum that is recovered by recycling. The information sources they use are bulletins posted on the web site of the Recycling Council of Ontario. Students are given data on the revenues generated from recycled aluminum cans and the average price of aluminum throughout the year. This enables them to determine the total mass of aluminum recovered. Once they are given the mass of a beverage can (see Exercise ❶ d), they can estimate the number of cans that were recycled. Using the fact that the total mass of aluminum cans sold each year in Ontario is about 30 000 000 kg, students can approximate the percentage of cans that are recycled. From this information, students can then draw some reasonable inferences about the amount of aluminum that is recovered through the recycling program. Activity 4 and this unit culminate in the requirement that the student consolidate the results of the investigations in Activities 2 through 4 in a report that compares the three methods mentioned above in their potential efficacy in meeting the present and future demands for beverage cans.

An important aspect of this activity is that it provides students with relevant as well as redundant data from which they must judiciously select to solve real-world problems. By accessing other Internet sites such as the Canadian Soft Drink Association (see web site on p. 87), students can gather their own data and create problems for others to solve.

Activity 4 – Student Page

Re-design, Reuse, or Recycle?

The following news release appeared on the web site of the Recycling Council of Ontario in December 1996.

http://www.web.net:80/rco/policy/csda_can.htm

Valuable Aluminum Cans to Stay, Soft Drink Makers Pledge to Recyclers

Toronto, December 17 -- To support Blue Box recycling, Ontario's major soft drink makers have committed to continue using high-value recyclable aluminum cans for the next three years, Canadian Soft Drink Association Vice President Stuart Hartley announced today.

…Province-wide, [recycled] aluminum is providing an estimated $14.8 million this year. This year the average market price for aluminum has been $1600 per tonne.

… Ontario consumers prefer to buy their soft drinks in cans. Nearly 90% of all soft drink containers purchased in Ontario are recyclable aluminum cans.

A later news update from the Recycling Council of Ontario appeared in early 1997.

http://www.rco.on.ca/news97/c2.htm

Soft Drink Container Skirmishes Continue

In its latest release on the issue, the Canadian Soft Drink Association is trying to squash rumours that it is orchestrating a switch from aluminum to plastic containers. According to the release, "aluminum containers currently account for 88% of the soft drink units sold in Ontario and more aluminum units were sold in 1996 than the year before."

…Aluminum Fast Facts

Aluminum soft drink cans sold in Ontario each year in kilograms: 30 million

The Lesson Launch
5 minutes

To launch this lesson, ask students questions such as:

- *What are two of the ways we have investigated to conserve our use of aluminum in the manufacture of beverage cans?*
- *Which of these two ways do you think would save more aluminum? Why?*
- *How does the recycling "blue box" program in Ontario conserve aluminum?*
- *Do you think recycling conserves much aluminum? What information would you need to decide?*

Through discussion, students should review the results of their investigations in Activities 2 and 3. Describe briefly the process of recycling. Mention that although aluminum cans are not pure aluminum, they are made from an aluminum alloy that is 97% aluminum. By gathering aluminum cans, melting them down and forming new aluminum alloy sheets that can be made into cans, we obviate the need to smelt more aluminum. This savings pays for the entire blue box program and the recycling of other products like paper and glass.

Individual Activity
30 minutes

Distribute copies of pages 86 and 87. Assign Exercises ❶ through ❹ on page 87 for students to work on individually. Remind students to check their notes from the previous activities to answer Exercises ❸ and ❹. They may need to confer with their previous partners if they recorded their results in only one notebook.

Students who have difficulty solving word problems will find Exercises ❶ and ❷ to be challenging. Avoid the temptation to give them the answers. Instead, respond to their questions by first ensuring that the student understands what is being asked. Then prompt the student with leading questions such as, "What information do you have that might help?", and "What additional information would you need to answer the question?" Keep the responsibility on the student to answer the question.

When students have finished the exercises, discuss the answers. Invite students who have answered the exercises successfully to display their solutions on the blackboard or overhead projector. This is a good time to reinforce writing complete statements to word problems.

Activity 4 – Student Page

❶ a) Do the two news bulletins on page 86 agree on the percent of soft drink containers sold in Ontario that are aluminum cans? Explain your answer.

b) How much revenue came from recycled aluminum cans in Ontario in 1996?

c) About how many tonnes of aluminum cans were recycled in Ontario in 1996? What is this mass in kilograms?

d) An aluminum can has a mass of about 13.6 g. About how many aluminum cans were recycled in Ontario in 1996?

❷ a) What was the total mass of aluminum in soft drink cans sold in Ontario in 1996?

b) About how many cans of soft drink were sold in Ontario in 1996?

c) What percent of the soft drink cans sold in Ontario in 1996 were recycled?

❸ Use the information in Activity 2 to estimate the percent of aluminum that could be saved if the wastage from the manufacturing process were retained and reused.

❹ Use the information you obtained in Activity 3 to estimate the percent of aluminum that could be saved if the dimensions of the standard beverage can were changed to a diameter of 8.0 cm and a height of 7.1 cm.

Research Report

Do you think we will eventually run out of aluminum for beverage cans? Write a report stating your opinion and provide reasons to support your position.

Address these questions in your report:

- How is the demand for aluminum cans expected to change in the 21st century?
- What are three measures that can be taken to conserve the supply of aluminum.
- How important is each of the conservation measures in preserving global aluminum resources?
- What measures would you recommend to ensure that sufficient aluminum will be available for beverage cans?

Web Sites for More Information

Recycling Council of Ontario: http://www.rco.on.ca
Canadian Soft Drink Association: http://www.softdrink.ca

Closure
20 minutes

Once it is clear that most students have reasonable estimates of the amount of aluminum that might be saved by each of the three measures mentioned above, assign the research report project on page 87. Encourage students to visit the cited web sites and others to obtain information for their reports. Attach a due date about seven days hence. This should provide enough time for all students to gain access to a computer and to print out their reports.

Re-design, Reuse, or Recycle?

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http://www.web.net:80/rco/policy/csda_can.htm

Valuable Aluminum Cans to Stay, Soft Drink Makers Pledge to Recyclers

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…Province-wide, [recycled] aluminum is providing an estimated $14.8 million this year. This year the average market price for aluminum has been $1600 per tonne.

…Ontario consumers prefer to buy their soft drinks in cans. Nearly 90% of all soft drink containers purchased in Ontario are recyclable aluminum cans.

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…Aluminum Fast Facts

Aluminum soft drink sold in cans in Ontario in 1996: 30 million kg.

❶ a) Do the two news bulletins on page 86 agree on the percent of soft drink containers sold in Ontario that are aluminum cans? Explain your answer.

b) How much revenue came from recycled aluminum cans in Ontario in 1996?

c) About how many tonnes of aluminum cans were recycled in Ontario in 1996? What is this mass in kilograms?

d) An aluminum can has a mass of about 13.6 g. About how many aluminum cans were recycled in Ontario in 1996?

❷ a) What was the total mass of aluminum in soft drink cans sold in Ontario in 1996?

b) About how many cans of soft drink were sold in Ontario in 1996?

c) What percent of the soft drink cans sold in Ontario in 1996 were recycled?

❸ Use the information in Activity 2 to estimate the percent of aluminum that could be saved if the wastage from the manufacturing process were retained and reused.

❹ Use the information you obtained in Activity 3 to estimate the percent of aluminum that could be saved if the dimensions of the standard beverage can were changed to a diameter of 8.0 cm and a height of 7.1 cm.

Research Report

Do you think we will eventually run out of aluminum for beverage cans? Write a report stating your opinion and provide reasons to support your position.

Address these questions in your report:

- How is the demand for aluminum cans expected to change in the 21st century?
- What are three measures that can be taken to conserve the supply of aluminum.
- How important is each of the conservation measures in preserving global aluminum resources?
- What measures would you recommend to ensure that sufficient aluminum will be available for beverage cans?

Web Sites for More Information

Recycling Council of Ontario: http://www.rco.on.ca
Canadian Soft Drink Association: http://www.softdrink.ca

Grade 8
Answer Key for Activity 4

❶ a) The bulletin at the top of the page states, “Nearly 90% of all soft drink containers purchased in Ontario are recyclable aluminum cans.” The bulletin below it states, “aluminum containers currently account for 88% of the soft drink units sold in Ontario…” The two statements are compatible because 88% is nearly 90%.

b) The revenue from recycled aluminum cans in Ontario in 1996 was $14 800 000.

c) The number of tonnes of recycled aluminum from cans was $14 800 000 ÷ 1600 or 9250 *t*.

d) The number of aluminum cans that were recycled in Ontario was:

$$\frac{9\,250\,000\,000}{13.6} \quad \text{or} \quad 680\,147\,058.$$

That is, about 680 000 000 cans were recycled in Ontario in 1996.

❷ a) The bulletin indicates that 30 000 000 kg of soft drink cans were sold in Ontario in 1996.

b) The number of cans required to have a total mass of 30 000 000 kg is 30 000 000 000 ÷ 13.6 or 2 205 882 353 cans.
That is, about 2 200 000 000 cans of soft drink were sold in Ontario in 1996.

c) The fraction of soft drink cans that were recycled in 1996 was 680 000 000 ÷ 2 200 000 000 or 0.309. That is, about 30% of the soft drink cans sold in 1996 were recycled.

❸ In Exercise ❸ of Activity 2 p. 71, it was stated that there is wastage of about 14% of the aluminum when the circles to make cans are stamped out of the aluminum alloy sheets. Therefore, about 14% could be saved by reusing the wastage in the manufacturing process.

❹ In the table in Exercise ❸ of Activity 3, p. 79, we discovered that the surface area of the standard beverage can is 284 cm^2 while the surface area of the shorter and fatter can is 279 cm^2. By changing to the shorter and fatter can, we would save about 5 cm^2 out of 284 cm^2 or about 1.8% of the aluminum used. By switching to the most economical shape (diameter:height = 1:1), we could save about 2.3%.

Research Report

The graph on page 62 suggests that the demand for beverage cans will expand significantly during the 21st century. This is supported by world-wide expansion of soft drinks into countries the previously had little or no access.

Students might be quite creative in the measures they propose for conserving the supply of aluminum. Among these are the following:

- mix aluminum with a cheaper metal to form an alloy.
- change the shape of the beverage can to a shorter and fatter cylinder.
- reuse the waste from the circle cut outs made during the manufacturing of the cans.
- encourage more people to recycle their beverage cans.

Through the exercises in this activity, students should have learned that changing the shape of the beverage can will yield savings of at most about 2%. Reusing the waste in the manufacturing process could save up to 14%. However, encouraging more people to recycle their aluminum cans could save the 70% of the aluminum that is currently lost as discarded, not recycled, beverage cans. The student report should indicate that encouraging more people to recycle their aluminum cans offers the greatest promise for the conservation of aluminum.

The scoring guide presented below has been developed using student responses on a field test conducted in 1998. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 4

	Level 1	Level 2	Level 3	Level 4
COMMUNICATION				
Comparison of the Relative Merits of Three Different Measures for Conserving the Aluminum in Beverage Cans. **General Expectations:** • Demonstrate a verbal and written understanding of and ability to apply accurate measurement and estimation strategies that relate to their environment. • Solve problems related to the calculation of the radius, diameter, and circumference of a circle.	• Report is mostly incoherent and contains two or fewer of the following elements. • observation that more beverage cans will be needed in the 21st century. • identification of three conservation measures: - reuse wastage - change shape of can - recycle • quantitative comparisons of the 3 conservation methods • reasonable recommendation with recycling identified as most important.	• Report is mostly coherent and contains two of the following elements. • observation that more beverage cans will be needed in the 21st century. • identification of three conservation measures: - reuse wastage - change shape of can - recycle • quantitative comparisons of the 3 conservation methods • reasonable recommendation with recycling identified as most important.	• Report is coherent and contains three of the following elements. • observation that more beverage cans will be needed in the 21st century. • identification of three conservation measures: - reuse wastage - change shape of can - recycle • quantitative comparisons of the 3 conservation methods • reasonable recommendation with recycling identified as most important.	• Report is coherent and contains all of the following elements. • observation that more beverage cans will be needed in the 21st century. • identification of three conservation measures: - reuse wastage - change shape of can - recycle • quantitative comparisons of the 3 conservation methods • reasonable recommendation with recycling identified as most important.

WHAT YOU MIGHT SEE

Level 1

Research Report

No, I don't think that we will
because in the garbage dumps
they seperate the different metals.
That is how they get metal for the
Cans. In the 21'st century cans are not
going to change. Do not throw in to
places where can not be retrieved. The
Conservation of Aluminum resources are
not important. I do not have any measures
to ensure that aluminum will be available.

This report is mostly incoherent. It fails to address the main issues called for and contains at most one of the elements listed in the scoring guide on page 89. The student does not seem to have understood the intent of the activities or the context of the investigations. An interview with the student would be needed to determine whether the report was an accurate assessment of the student's comprehension of the issues.

Level 2

Reaserch Report

I think as the population is growing so is
the demand for aluminum soft drink cans.
Manufacturers should start to use the material
that is left over ~~as~~ after the can is made.
After all 21.5% of all aluminum is
wasted. We should also warn consumers that
if we ~~do~~ not recycle all the cans we can,
one day we will simply run out. Manufactures
should also consider the enviroment because
aluminum is harmful to the soil and forest
and if it is thrown on the ground and not
picked up it will take years for it to
decompose.

← The report indicates that more beverage cans will be needed in the 21st century.

← Reusing wastage is identified as a conservation measure and some data is given to quantify its potential.

← Recycling beverage cans is included among the recommended ways to conserve aluminum .

This report is mostly coherent. It includes two of the four elements listed in the scoring guide on page 89, i.e., the first and last. It also identifies reusing wastage and recycling cans as two measures to conserve aluminum but it does not compare the relative potential of these measures for saving aluminum. The student supports the recycling recommendation with the additional argument that aluminum is not biodegradable and must be recycled to preserve the environment.

WHAT YOU MIGHT SEE

Level 2

RESEARCH REPORT

I don't think we'll run out of aluminum beverage cans for a long time if people continue to recycle. However, if the demand increases at a drastic rate, which might be possible, we will, eventually run out, but even if we do, it won't be for a while.

There are certain precautions we can take so we don't run out, even though the demand for aluminum beverage cans will most likely increase throughout the 21st century.

← The report indicates that more beverage cans will be needed in the 21st century.

Maybe companies should start making taller and thinner cans, which are more economical, and that people should recycle more. Surprisingly, only 32% of the aluminum cans get recycled which I think is very little.

← Changing the height-to-diameter ratio is correctly identified as a conserving measure, but the student has recommended increasing rather than decreasing the height to diameter ratio.

← Recycling is identified as a conservation measure and some data is given to suggest its potential savings.

Each one of these measures is important because they can help the environment and also the soft drink industry, which might lose quite a bit of money if it was to run out of aluminum for beverage cans, since about 90% of beverage containers purchased in Ontario are made of aluminum, which suggests that a lot of people prefer aluminum to plastic, glass or other materials.

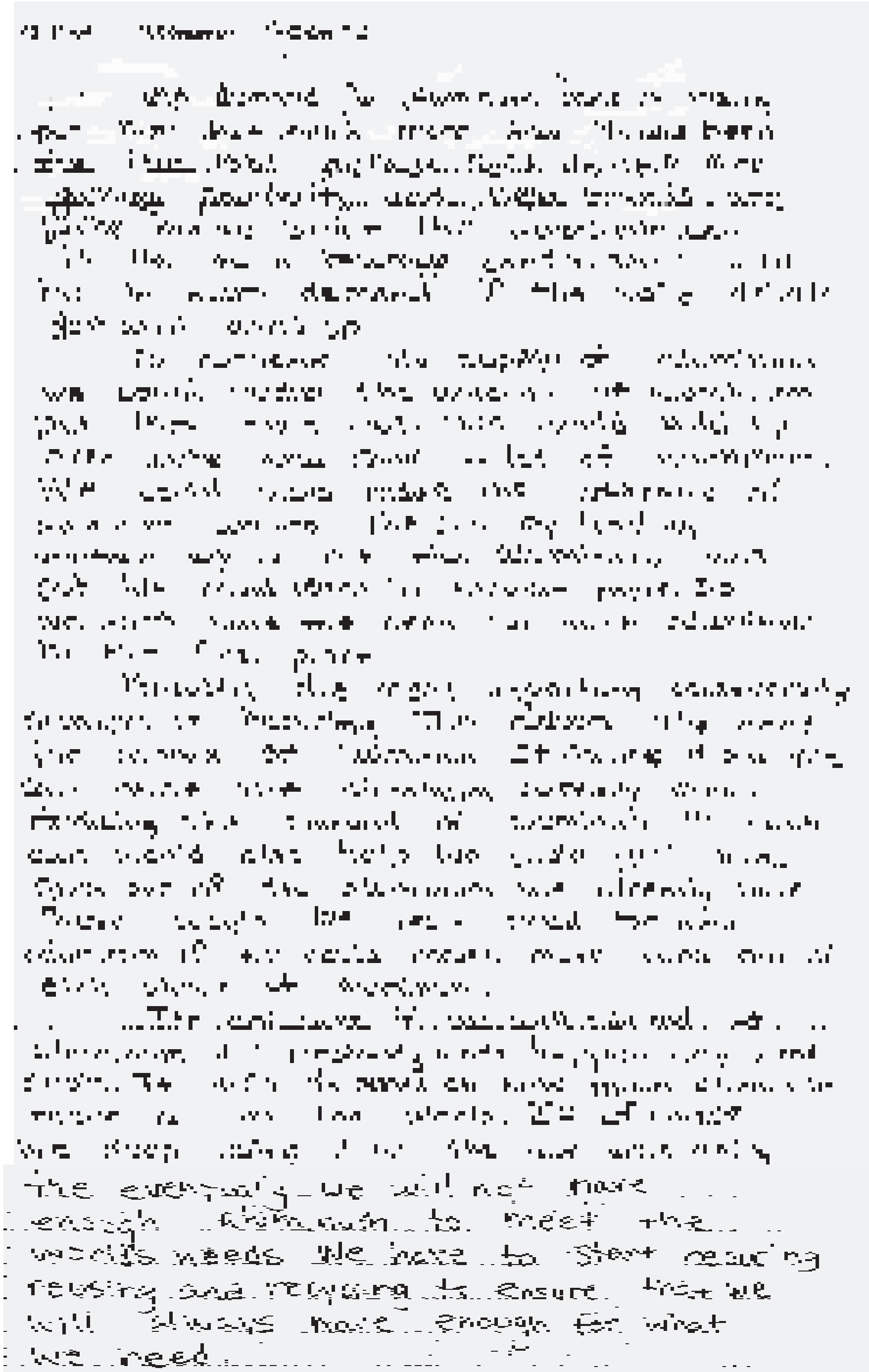
← Recycling beverage cans is included among the recommended ways to conserve aluminum .

If we all take these measures, we will have aluminum beverage cans around for a long time.

This report is coherent and contains two of the four elements listed in the scoring guide on page 89, i.e., the first and last. It also includes two of the three conservation measures and provides some quantitative data. The report suggests that the two conservation measures mentioned are both important in conserving the available aluminum, but makes no comparison about the relative merits of each measure. Inclusion of the reuse of the wastage and some comparative observations about the relative importance of each measure would move this to a Level 3.

WHAT YOU MIGHT SEE

Level 3



← The report indicates that more beverage cans will be needed in the 21st century.

← Using less aluminum per can is identified, although changing the height-to-diameter ratio is not explicitly mentioned.

← Reusing the wastage is identified as a conservation measure.

← Recycling is identified as a conservation measure.

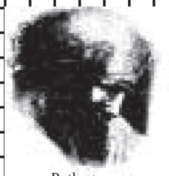
← Recycling is identified as the conservation measure that has the greatest potential.

← Recycling beverage cans is included among the recommended ways to conserve aluminum .

The report is coherent and well organized. It addresses the main questions posed and reflects a good understanding of the issues involved in the conservation of aluminum beverage cans. It contains three of the four elements listed in the scoring guide on page 89. The only missing element is the quantitative data needed to compare the potential effectiveness of the three measures. Changing the height-to-diameter ratio of the can could save up to 2% and reusing the wastage could save up to 14%, but recycling could save another 70% above current levels. It is the large magnitude of this measure relative to the others that represents a compelling argument for recycling as the most important measure.

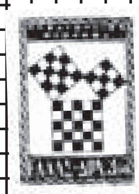
Answer Key for Templates on Pages 15–18

15 The Pythagorean Relationship

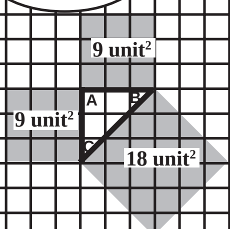


Pythagoras
582 B.C – 501 B.C.

The Pythagorean Theorem may be the most important theorem in geometry. It is important that students have lots of opportunity to measure the lengths of sides of right triangles to verify it and to internalize its meaning.

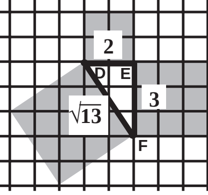


The areas of the squares are shown below.



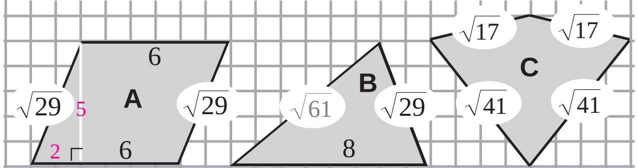
To calculate the length |BC| we write:
$|BC|^2 = 9 + 9 = 18.$
$|BC| = \sqrt{18} \approx 4.24$

To calculate the length |DF| we write:
$|DE| = 2$ and $|EF| = 3$,
so $|DF|^2 = 2^2 + 3^2 = 13.$
$|DF| = \sqrt{13} \approx 3.60.$



16 Using The Pythagorean Relationship To Calculate Perimeter

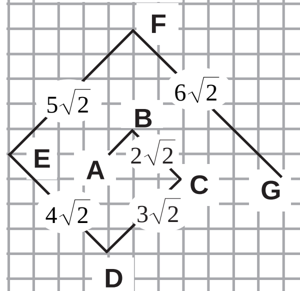
❶ The lengths of the sides and the perimeters of the shapes are shown below.



Perimeter
$= 2(6 + \sqrt{29})$
≈ 22.77 units.

Perimeter
$= 8 + \sqrt{61} + \sqrt{29}$
≈ 21.20 units.

Perimeter
$= 2(\sqrt{17} + \sqrt{41})$
≈ 21.05 units.

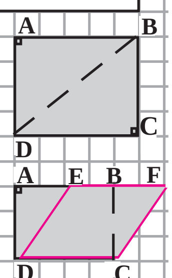


❷ A quick way to estimate the length of the path is to observe that the diagonal of a unit square is $\sqrt{2}$ units or about 1.4 units long. The arms of the spiral path are respectively 1, 2, 3, 4, 5, and 6 of these diagonals long. The total length is:

$(1 + 2 + \ldots + 6)\sqrt{2}$ or about $21 \times 1.4 = 29.4$ units. The exact length is $21\sqrt{2}$ or about 29.7 units.

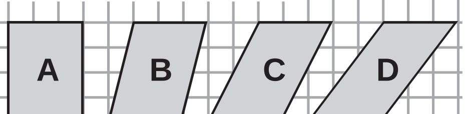
17 Relating the Area of a Parallelogram to the Area of a Rectangle

❶ ΔABD and ΔCDB are congruent. Reasons given will vary, but any reasonable explanation or set of sufficient conditions should be accepted. Since the triangles are congruent, they have equal area. It is half the area of rectangle ABCD, i.e., 10 unit². To find the area of a right triangle, we can multiply the lengths of its shorter sides and divide the product by two.



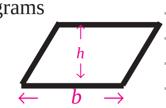
❷ Students should discover that the areas of ABCD and EFCD are equal.

Shape	Base	Height	Area
A	3	4	12
B	3	4	12
C	3	4	12
D	3	4	12



❸ As shown in the table, all the parallelograms have the same base, height and area.

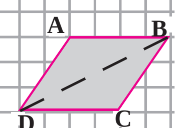
❹ a) The area of a parallelogram is the product of its base and height.
b) Area = $b \times h$



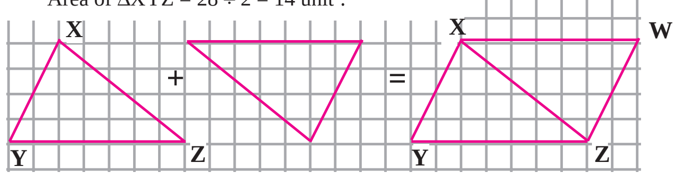
The area of a parallelogram can be calculated by multiplying its length by its width only if it's a rectangle.

18 Relating the Area of a Triangle to the Area of a Parallelogram

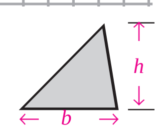
❶ See answer to Exercise ❶ p. 17.
ABCD has area 12 unit². ΔABD and ΔCBD each have an area of 6 unit² because each is half the area of the parallelogram.



❷ Area of parallelogram XYZW is $7 \times 4 = 28$ unit².
Area of $\Delta XYZ = 28 \div 2 = 14$ unit².

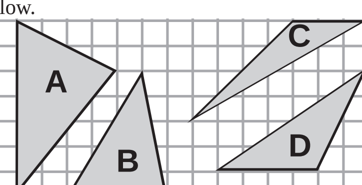


❸ a) Divide the product of the base and height by 2.
b) $A = (b \times h)/2$



❹ Students may need help determining which side of a triangle is the base. Show them that the height of a triangle on a grid is either horizontal or vertical. A base is a side perpendicular to the height. A base, height, and the area of each triangle are given in the table below.

Triangle	Base	Height	Area
A	7	4	14
B	4	5	10
C	3	4	6
D	4	4	8



TEMPLATE

Record of Student Achievement on the Grade 7 Unit

Is It Mathematics or Magic?

Student Name ____

From Scoring Guide for Activity 1 p. 27

Topic	Level
Concepts	
Application	

From Scoring Guide for Activity 2 p. 35

Topic	Level
Problem Solving	
Concepts	

From Scoring Guide for Activity 3 p. 43

Topic	Level
Application	
Communication	

From Scoring Guide for Activity 4 p. 51

Topic	Level
Problem Solving	
Communication	

Combining the Scores from all Scoring Guides

Summary	Level
→ Problem Solving	
→ Concepts	
→ Application	
→ Communication	

Student Name ____

From Scoring Guide for Activity 1 p. 27

Topic	Level
Concepts	
Application	

From Scoring Guide for Activity 2 p. 35

Topic	Level
Problem Solving	
Concepts	

From Scoring Guide for Activity 3 p. 43

Topic	Level
Application	
Communication	

From Scoring Guide for Activity 4 p. 51

Topic	Level
Problem Solving	
Communication	

Combining the Scores from all Scoring Guides

Summary	Level
→ Problem Solving	
→ Concepts	
→ Application	
→ Communication	

Student Name ____

From Scoring Guide for Activity 1 p. 27

Topic	Level
Concepts	
Application	

From Scoring Guide for Activity 2 p. 35

Topic	Level
Problem Solving	
Concepts	

From Scoring Guide for Activity 3 p. 43

Topic	Level
Application	
Communication	

From Scoring Guide for Activity 4 p. 51

Topic	Level
Problem Solving	
Communication	

Combining the Scores from all Scoring Guides

Summary	Level
→ Problem Solving	
→ Concepts	
→ Application	
→ Communication	

Student Name ____

From Scoring Guide for Activity 1 p. 27

Topic	Level
Concepts	
Application	

From Scoring Guide for Activity 2 p. 35

Topic	Level
Problem Solving	
Concepts	

From Scoring Guide for Activity 3 p. 43

Topic	Level
Application	
Communication	

From Scoring Guide for Activity 4 p. 51

Topic	Level
Problem Solving	
Communication	

Combining the Scores from all Scoring Guides

Summary	Level
→ Problem Solving	
→ Concepts	
→ Application	
→ Communication	

Record of Student Achievement on the Grade 8 Unit

The Ingenious Beverage Can – Will We Ever Run Out?

Student Name ____

From Scoring Guide for Activity 1 p. 65

Topic	Level
Concepts	

From Scoring Guide for Activity 2 p. 73

Topic	Level
Problem Solving	
Concepts	

From Scoring Guide for Activity 3 p. 81

Topic	Level
Application	

From Scoring Guide for Activity 4 p. 89

Topic	Level
Communication	

Combining the Scores from all Scoring Guides

Summary	Level
Problem Solving	
Concepts	
Application	
Communication	

Student Name ____

From Scoring Guide for Activity 1 p. 65

Topic	Level
Concepts	

From Scoring Guide for Activity 2 p. 73

Topic	Level
Problem Solving	
Concepts	

From Scoring Guide for Activity 3 p. 81

Topic	Level
Application	

From Scoring Guide for Activity 4 p. 89

Topic	Level
Communication	

Combining the Scores from all Scoring Guides

Summary	Level
Problem Solving	
Concepts	
Application	
Communication	

Student Name ____

From Scoring Guide for Activity 1 p. 65

Topic	Level
Concepts	

From Scoring Guide for Activity 2 p. 73

Topic	Level
Problem Solving	
Concepts	

From Scoring Guide for Activity 3 p. 81

Topic	Level
Application	

From Scoring Guide for Activity 4 p. 89

Topic	Level
Communication	

Combining the Scores from all Scoring Guides

Summary	Level
Problem Solving	
Concepts	
Application	
Communication	

Student Name ____

From Scoring Guide for Activity 1 p. 65

Topic	Level
Concepts	

From Scoring Guide for Activity 2 p. 73

Topic	Level
Problem Solving	
Concepts	

From Scoring Guide for Activity 3 p. 81

Topic	Level
Application	

From Scoring Guide for Activity 4 p. 89

Topic	Level
Communication	

Combining the Scores from all Scoring Guides

Summary	Level
Problem Solving	
Concepts	
Application	
Communication	

Additional Resources for Measurement

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Duke, Charlotte. "Tangrams and Area." *Mathematics Teaching in the Middle School*, Vol 3, #7. Reston, VA: National Council of Teachers of Mathematics, May, 1998, pp. 485-487.

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Hayes, Nakonia. "Cardboard Tubes: Bring Geometry from Home." *Mathematics Teaching in the Middle School,* Vol 4, #2. Reston, VA: National Council of Teachers of Mathematics, October, 1998, pp. 120-122.

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Verderber, Nadine L. "Tin Cans Revisited." *Mathematics Teacher,* Vol 85. Reston, VA: National Council of Teachers of Mathematics, May, 1992, pp. 346-349.

Web Sites

Centre for Innovation in Mathematics Teaching	http://www.ex.ac.uk/cimt/
Eisenhower National Clearinghouse	http://www.enc.org/
Measurement Converter Activity	http://www.mplik.ru/~sg/transl/
NRICH MATHS–the On-line Maths Club	http://nrich.maths.org.uk/
TI Calculator Programs and Activities	http://www.ti.com/calc/docs/resource.htm

Free Software for Ontario Schools

The Ministry of Education and Training of Ontario purchases site licences of software for all publically funded schools in the province. This software can be obtained from the Ontario Educational Software Service (OESS) representative in your school district. To determine what is available, access this web site: **http://www.tvo.org/osapac**